AN INVESTIGATION OF THE STRONGNESS PROPERTY FOR NEARNESS FRAMES

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MIRJAM NANGOMBE HAIMENE

200848852

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Main supervisor: Dr M.M. Mugochi, University of Namibia

Abstract

This study will investigate the strongness property for nearness and nearness partial frames. We initially revisit the concepts of strong and totally strong nearness frame and study their closures under completion. We also explore the properties of totally strong and uniformly completely regular nearness frames, and study the relationship between them. We show that the category of totally strong nearness frames is coreflective in the category of uniformly completely regular nearness frames. This we do by constructing the coreflection. Further, we study nearness in the context of partial frames with particular emphasis to the strong and totally strong properties. We follow [6] in constructing the respective coreflections using the notion of *P*-Approximations.

Keywords: Nearness frames, regular frames, strong, totally strong, uniformly completely regular, interpolation, almost uniform, partial frames, uniform, meet-semilattice, *P*-approximation, uniformity, coreflection.

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Glossary

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RegFrm	Regular frames	Frame homomorphisms	**??**
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TStrNFrm	Totally strong nearness frames	Uniform frame homomorphisms	13
AuNFrm	Almost uniform nearness frames	Uniform frame homomorphisms	14
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Declaration

I, Mirjam Haimene, hereby declare that this study, **An investigation of the strongness property for nearness frames**, is my own work and is a true reflection of my research, and that this work, or any part thereof has not been submitted for a degree at any other institution.

Ms Mirjam Haimene Signature: .... March 2018

Chapter 1

Introduction

1.1 Background of the study

The idea of a frame was introduced in the mid-sixties by Dowker and Papert [17] and the theory of nearness was introduced by H. Herrlich [13] in 1972, as an axiomatization of the concept of nearness between arbitrary collections of sets. Since then frame and nearness have been widely studied by many mathematicians. Banaschewski and Pultr have introduced the concept of nearness frames and then constructed completions of nearness frames [2] which give rise to a coreflection for strong nearness frames [3].
The theory of coreflections in categories serves the purpose of unifying various fundamental constructions in mathematics, through "universal" properties that each possesses. The purpose of this thesis is to study certain categorical constructions among nearness frames. It was proved in [15, 16] that totally strong nearness frames are closed under completion. In this study we show that totally strong nearness frames are coreflective in the category of uniformly completely regular nearness frames.
Further, we study partial frames, which are meet-semilattices in which certain joins exist and finite meets distribute over these joins. These joins will be specified by means of a selection function. While any selection function must fulfill certain axioms to yield a feasible theory, the axioms used in this study were all introduced in [6].
In this study we investigate properties of nearness partial frames. These have been introduced in [6, 7] to provide a fertile context in which to do pointfree structured and unstructured topology, using a small collection of axioms of an elementary nature. In [6], J. Frith and A. Schauerte presented axioms for partial frames and established a

method for constructing a coreflection for all nearness partial frames and, in [7] they provided a coreflection for uniform partial frames.

We next investigate whether the completion is a coreflection for totally strong nearness partial frames. In full (that is, not partial) frames, the natural tool used to determine a coreflection is the existence of a right adjoint of frame maps, but in $\mathcal{S}$-frames these are, in general, not available since, for instance, $\mathcal{S}$-frame maps need not preserve arbitrary joins. Therefore we will follow [6] in constructing the coreflection for totally strong nearness partial frames using the notion of P-Approximations.

1.2 Outline of the study

The study is organized as follows. Chapter 1 begins with a brief account of the necessary background and terminology on frames and nearness frames, as well as a literature review, which includes references that are crucial for our study. In Chapter 2, we present the relevant definitions pertaining to frames, nearness frames and uniform frames, and outline the necessary background for the subsequent chapters. A general reference for categorical concepts used in the thesis is [14], for frames it is [18] and for a pioniering article on nearness frames refer to [2]. In Chapter 3, we introduce strong and totally strong nearness frames and investigate the relationship between them [15]. We also show that the category of totally strong nearness frames is closed under completion [15]. Further, we construct a coreflection for totally strong nearness frame in the category of uniformly completely regular nearness frames .

Chapter 4 is essentially an introduction to partial frames and the structure of a nearness partial frame. Here we extend the relevant definitions relating to partial frames and nearness partial frames, we consider the category of totally strong nearness partial frames and try to examine it is coreflectivity in the category of strong nearness partial frames.

1.3 Literature review

The concept of frame was discovered in 1966 by C.H. Dowker and brought to the fore by D. Papert [4, 11]. H. Herrlich in 1972 introduced the idea of a nearness while B.

Banaschewski (based on joint work with A. Pultr) in 1990 announced the concept of nearness frames in a series of lectures delivered at the University of Cape Town [2]. When nearness frames were introduced, several questions naturally arising from the unique existence of the completion of a nearness frame were scrutinized. In [2] it was established that every nearness frame has a completion. This result was then extended to a coreflection for uniform frames. Banaschewski, Hong and Pultr [3] investigate the existence of a coreflection for uniform frames and proved that completion is not a coreflection on **NFrm**, but it is on the subcategory **StrNFrm** of strong nearness frames, a subcategory strictly containing almost uniform nearness frames. Thereafter, Dube and Mugochi [5] revisited the work done in [3] and found that these (almost uniform) nearness frames are coreflective in the category of all nearness frames with interpolating uniformly below relation.

Further, Mugochi [15] investigated quotient-fine nearness frames, showing that they are reflective in the category of strong nearness frames and also in those with spatial completion. Besides he established that the category of totally strong nearness frames is closed under completion. Frith and Schauerte [8] examined a general method of constructing coreflections in the category of nearness frames. Their study provided a way to establish that strong coreflections can change the underlying frame in nearness frames, in contrast with the work done in [5]. We aim to construct a coreflections in the category of totally strong nearness frames.

Due to the lack of joins in some subsets of partial frames (unlike in full frames), different authors have used different sets of axioms for their selection functions to complete their investigations. A small collection of axioms of an elementary nature allows one to do much traditional pointfree topology, both on the level of partial frames and that of uniform partial frames. Frith and Schauerte [6] presented axioms for partial frames and established a method for determining certain coreflective subcategories of nearness partial frames. Moreover, the same authors [7] constructed a completion and a Samuel compactification for uniform partial frames.

In this study we focus on the subcategory of totally strong nearness frames and totally strong nearness partial frames and attempt to establish their coreflectivity in the categories of uniformly completely regular nearness frames and nearness partial frames, respectively.

Chapter 2

Preliminaries on frames

In this introductory section we collect a few facts that will be relevant for our study, and fix notation. We begin by giving definitions pertaining to frames and nearness frames. For general theory of frames we refer to [12, 18], and for nearness frames we refer to [2].

2.1 Relations and Lattices

We first introduce the following notions on relations and lattices. By a partial order $\leq$ on a set L is meant a binary relation on L (i.e $\leq$ is a subset of $L \times L$) satisfying the following properties: For all $a, b, c \in L$:

(i) reflexivity: $a \leq a$.

(ii) transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$.

(iii) antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$.

If L is equipped with a partial order we call L a partially ordered set or refer to the pair $(L, \leq)$ as a *poset*. Let $(L, \leq)$ be a poset and $S \subseteq L$. Then

(i) $0 \in L$ is called the *minimum* (or *bottom*) element of L if and only if for all $a \in L$, $0 \leq a$.

(ii) $1 \in L$ is called the *maximum* (or *top*) element of L if and only if for all $a \in L$, $a \leq 1$.

(iii) $a \in L$ is called a *lower bound* of S if and only if for all $s \in S$, $a \leq s$. If a is a lower bound of S and $a \in S$, then a is the *minimum* of S.

(iv) $b \in L$ is called an *upper bound* of S if and only if for all $s \in S$, $s \leq b$. If this $b \in S$, then b is the *maximum* of S.

(v) $\alpha \in L$ is called the *greatest lower bound* (or *meet*) of S if and only if α is a lower bound of S and $a \leq \alpha$ for every lower bound a of S.

(vi) $\beta \in L$ is called the *least upper bound* (or *join*) of S if and only if β is an upper bound of S and $\beta \leq b$ for every upper bound b of S.

We use the notation $\bigwedge S$ for the meet of S and the notation $\bigvee S$ for the join of S. When $S = \{a,b\}$, we write $\bigwedge\{a,b\} = a \wedge b$ and is read as " a meet b", and $\bigvee\{a,b\} = a \vee b$ and read as " a join b". A *lattice* is a poset with a top and a bottom element such that every pair of it is elements has a meet and a join. A *complete lattice* is a poset in which any subset has a meet and a join.

2.2 Frames and Nearness frames

This section will include some basic definitions on frames and nearness frames bearing relevance to our study.

Definition 2.2.1. (a) A *frame* L is a complete lattice satisfying the infinite distributive law: for any $a \in L$ and any $S \subseteq L$, $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$.

(b) A subset $M \subseteq L$ is called a *subframe* of L if it is closed under finite meets and arbitrary joins restricted to L.

Definition 2.2.2. (a) A *frame homomorphism* (or *frame map*) between frames L and M is a map $h : L \to M$ which preserves finite meets and arbitrary joins. In consequence $h(0) = 0$ and $h(1) = 1$.

(b) To any frame homomorphism $h : L \to M$, the map $h_* : M \to L$, defined by $h_*(y) = \bigvee\{x \in L \mid h(x) \leq y\}$ is categorically the *right adjoint* of h, which is not necessarily a frame homomorphism, but preserves arbitrary meets. The following property holds for every $x \in L$ and every $y \in M$:

$$h(x) \leq y \Longleftrightarrow x \leq h_*(y).$$

(c) A frame homomorphism $h : L \to M$ is *dense* if for every $a \in L$, $h(a) = 0$ implies $a = 0$. This holds if and only if $h_*(0) = 0$.

Remark 2.2.3. A frame homomorphism $h : L \to M$ is onto if and only if $h \circ h_* = id_M$.

To see this, first suppose h is onto. Let $b \in M$ and $a \in L$ be such that $h(a) = b$. Then,

$$\begin{aligned} hh_*(b) &= h(h_*(b)) \\ &= h(\bigvee\{s \in L \mid h(s) \leqslant b\}) \\ &= h(\bigvee\{s \in L \mid h(s) \leqslant h(a)\}) \\ &= \bigvee\{h(s) \mid h(s) \leqslant h(a)\}, \text{since } h \text{ preserves joins} \\ &\leqslant h(a) = b. \end{aligned}$$

So,

$$hh_*(b) \leqslant b. \tag{1}$$

Now, by definition $c \leqslant h_*h(c)$ for any $c \in L$. So, $a \leqslant h_*h(a)$ in particular. Hence $b = h(a) \leqslant h(h_*(h(a))) = hh_*(b)$. Therefore,

$$b \leqslant hh_*(b). \tag{2}$$

Hence from (1) and (2) $hh_*(b) = b$.
Conversely, suppose $hh_* = id$. Let $b \in M$. So, $hh_*(b) = b$. Take $a = h_*(b) \in L$. Hence h is onto.

We write **Frm** for the category of frames and frame homomorphisms. In our discussions that follow, L will normally be a frame unless otherwise specified.

Definition 2.2.4. (a) By a *cover* of L is meant a subset $G \subseteq L$ whose join equal to the top element (i.e $\bigvee G = 1$). The set of all covers of L is denoted by $\mathrm{Cov}(L)$.

(b) For covers C and D of L, C is said to *refine* D, written $C \leq D$, if for every $c \in C$ there exists $d \in D$ such that $c \leq d$. For any $A \in \mathrm{Cov}(L)$ and $x \in L$, the element Ax of L is defined by $Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}$ and is called *the star of x relative to A*. Notice that $x \leqslant Ax$. If $A, B \in \mathrm{Cov}(L)$, then $AB = \{Ab \mid b \in B\}$ is also a cover. To see this, notice that $\bigvee AB = \bigvee\{Ab \mid b \in B\} = \bigvee_{b\in B} Ab \geq \bigvee_{b\in B} b = \bigvee B = 1$.

(c) A cover A *star-refines* a cover B, written $A \leq^* B$, if $AA \leq B$.

(d) Given a collection $\mathcal{N} \subseteq \mathrm{Cov}(L)$, we say $x \in L$ *is* $\mathcal{N}$*-strongly below* $y \in L$, written $x \lhd_{\mathcal{N}} y$ if there is a cover $A \in \mathcal{N}$ such that $Ax \leqslant y$.

Definition 2.2.5. (a) For any element $a \in L$ the sets:

$$\downarrow a = \{x \in L \mid x \leqslant a\} \text{ and} \uparrow a = \{x \in L \mid a \leqslant x\}$$

are called the *downset* at a and the *upset* at a, respectively.

(b) A subset $D \subseteq L$ is called a *lower set* (or *downset*) if, whenever $x \in D$ and $y \leqslant x$, then $y \in D$.

(c) A subset $U \subseteq L$ is called an *upper set* (or *upset*) if, whenever $x \in U$ and $x \leqslant y$, then $y \in U$.

(d) A subset $S \subseteq L$ is said to *generate* L if for every element $x \in L$, we have $x = \bigvee\{s \in S \mid s \leqslant x\}$.

(e) If L has finite meets, a subset F of L is called a *filter* of L if it is an upper set closed under finite meets.

We now define the terms nearness and nearness frames.

Definition 2.2.6. (a) A nonempty collection $\mathcal{N}L \subseteq \mathrm{Cov}(L)$ is called a nearness on L if the following hold:

n1. Whenever $A \in \mathcal{N}L$ refines $B \in \mathrm{Cov}(L)$, then $B \in \mathcal{N}L$.

n2. Whenever $A, B \in \mathcal{N}L$, then $A \wedge B = \{a \wedge b \mid a \in A, b \in B\} \in \mathcal{N}L$.

n3. Every $x \in L$ can be expressed as $x = \bigvee\{y \in L | y \lhd_{\mathcal{N}} x\}$. This property is referred to as the *admissibility property*.

(b) The pair $(L, \mathcal{N}L)$ is called a *nearness frame*.

(c) A nearness $\mathcal{N}L$ on a frame L is said to be *generated* by $\mathcal{C} \subseteq \mathcal{N}L$ if for every $A \in \mathcal{N}L$ there exists $C \in \mathcal{C}$ such that $C \leqslant A$. Where $\mathcal{C}$ is the collection of covers.

In the case where $\mathcal{N}L$ is a nearness on L, we refer to the relation $\lhd_{\mathcal{N}L}$ (or simply $\lhd$) as the *uniformly below relation* on L. For covers A and B of a nearness frame, we say A is uniformly below B and write $A \lhd B$ to mean that for every $a \in A$ there is $b \in B$ such that $a \lhd b$.

Definition 2.2.7. Let $(L, \mathcal{N}L)$ and $(M, \mathcal{N}M)$ be nearness frames,

(a) The members of $\mathcal{N}L$ are called *uniform covers* of L.

(b) If $C \in \mathrm{Cov}(L)$, $\check{C}$ is the cover defined by $\check{C} = \{x \in L \mid x \lhd c \text{ for some } c \in C\}$.

(c) A frame map $f : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ is a *uniform map* if for every $C \in \mathcal{N}L$, $f[C] = \{f(c) \mid c \in C\} \in \mathcal{N}M$.

(d) $(L, \mathcal{N}L)$ is *strong* if whenever $C \in \mathcal{N}L$, then $\check{C} \in \mathcal{N}L$, (notice that $\check{C} \lhd C$). Equivalently, $(L, \mathcal{N}L)$ is *strong* if for every uniform cover A, there is a uniform cover B such that $B \lhd A$.

(e) $(L, \mathcal{N}L)$ is *almost uniform* if $(L, \mathcal{N}L)$ is strong and the relation $\lhd$ interpolates in L, i.e for any $x, y \in L$, if $x \lhd y$, then there is $z \in L$ such that $x \lhd z \lhd y$.

(f) A nearness is called a *uniformity* if every uniform cover has a uniform star refinement. The pair $(L, \mathcal{N}L)$ is a *uniform frame* if $\mathcal{N}L$ is a uniformity on L.

Denote by **NFrm** the category of nearness frames and uniform frame homomorphisms. The following two results appear in [3] and here we provide the proofs.

Lemma 2.2.8. *Suppose $(L, \mathcal{N}L)$ is a nearness frame. Let $B \leqslant^* C$ in $\mathcal{N}L$. Then*

(a) $B \subseteq \check{C}$

(b) if $Cx \leqslant y$ (i.e $x \lhd y$), then $B(Bx) \leqslant y$.

Proof. (a) Since $B \leqslant^* C$, we have that $BB \leqslant C$. Now $BB = \{Bb \mid b \in B\}$. So for all $b \in B$, let $c \in C$ be such that $Bb \leqslant c$. This implies that for all $b \in B$, there is $c \in C$ such that $b \lhd c$. Therefore $B \subseteq \check{C}$. (Notice also that this means $B \lhd C$).
(b) Suppose $Cx \leqslant y$, i.e $x \lhd y$. Then, since $BB \leqslant C$ (so that $(BB)x \leqslant Cx$), we have that $(BB)x \leqslant y$. Thus $B(Bx) \leqslant y$. □

Proposition 2.2.9. *If $(L, \mathcal{N}L)$ is a uniform frame, then it is almost uniform.*

Proof. First, to see that $\mathcal{N}L$ is strong, Let $B, C \in \mathcal{N}L$ be such that $B \leqslant^* C$. Then, by Lemma 2.2.8, $B \subseteq \check{C}$. So $\check{C} \in \mathcal{N}L$. Therefore $\mathcal{N}L$ is strong.
Second, to see that $\lhd$ interpolates in L, suppose $x \lhd y$ in L. Let $B, C \in \mathcal{N}L$ be such that $B \leqslant^* C$ and $Cx \leqslant y$. Then $BBx \leqslant Cx \leqslant y$. Also, since $x \leqslant Bx$, we have $Bx \leqslant BBx$. Put $z = BBx$. Then $x \lhd z$. So, we conclude that $x \lhd z \lhd y$, as desired. □

Proposition 2.2.10. *The following properties hold for the relation $\lhd$:*

(a) If $x \lhd y$, $a \leqslant x$ and $y \leqslant b$, then $a \lhd b$.

(b) If $x \lhd y$ and $a \lhd b$, then $x \wedge a \lhd y \wedge b$ and $x \vee a \lhd y \vee b$.

(c) If $\mathcal{N}L$ is a uniformity, then $x \lhd y$ implies $x \lhd z \lhd y$ for some $z \in L$.

Proof. To show (a) suppose $x \lhd y$, $a \leqslant x$ and $y \leqslant b$. Then we have $a \leqslant x \lhd y \leqslant b$. So, $a \lhd y \leqslant b$. Thus $a \lhd b$.
To prove (b), suppose $x \lhd y$ and $a \lhd b$ and let $A \in \mathcal{N}L$. Then by hypothesis, $Ax \leqslant y$ and $Aa \leqslant b$ respectively. So $A(x \wedge a) = Ax \wedge Ay \leqslant y \wedge b$, as required. In the same way one can show $x \vee a \lhd y \vee b$.
(c) By Proposition 2.2.9, if $\mathcal{N}L$ is uniformity, then $\mathcal{N}L$ interpolates. So for $x \lhd y$, there exists $z \in L$, such that $x \lhd z \lhd y$. □

Concerning nearness and onto frame homomorphisms, we have the following definitions as stated in [16].

Definition 2.2.11. A homomorphism $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ between the respective underlying frames is said to be:

(a) a *surjection, or quotient map* if it is onto and $\mathcal{N}M = \{h[C] \mid C \in \mathcal{N}L\}$. In this case we refer to the nearness frame $(M, \mathcal{N}M)$ as a *quotient* of $(L, \mathcal{N}L)$.

(b) a *strict surjection* if it is a dense surjection and the uniform covers $h_*[C]$, $C \in \mathcal{N}M$, generate $\mathcal{N}L$.

The notion of regularity in frames plays a major role in the theory of nearness frames. The *rather below* $\prec$ relation on a frame L is defined by: $y \prec x$ if and only if there is $z \in L$ (called a *separating element*) such that $y \wedge z = 0$ and $x \vee z = 1$. Note that $y \prec x$ is also expressed by the condition that $x \vee y^* = 1$ where $y^* = \bigvee\{a \in L \mid a \wedge y = 0\}$ is the *pseudocomplement* of y.

It is necessary to note that if $x \lhd y$ in any nearness frame L, then $y^* \lhd x^*$. This is so because if $x \lhd y$ then $\{x^*, y\}$ is a uniform cover (to see this let $C \in \mathcal{N}L$ be such that $Cx \leqslant y$. Since $x \leqslant Cx \leqslant y$, then $C \leqslant \{x^*, y\} \in \mathcal{N}L$, also $y \vee x^* = 1$) and $\{x^*, y\}y^* = x^*$.

Definition 2.2.12. A frame L is said to be *regular* if every $x \in L$ is expressible as

$$x = \bigvee\{y \in L \mid y \prec x\}.$$

It is well known (see [2]) that a frame has a nearness if and only if it is regular. Henceforth we shall assume all our frames to be regular unless otherwise specified.

Chapter 3

Strong and totally strong nearness frames

3.1 Strong nearness frames

Strong nearness frames were introduced in [2]. They play a vital role in the study of completions of nearness frames. In this section we recall some properties of these nearness frames.

We write **StrNFrm** for the category of strong nearness frames and uniform frame homomorphisms. The next results appear in [1], and we state without proof.

Lemma 3.1.1. *(a) If $(L, \mathcal{N}L)$ is a strong nearness frame, then any dense surjection $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ is strict.*

(b) A quotient of a strong nearness frame is strong.

(c) If $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ is a dense surjection, then $(L, \mathcal{N}L)$ is strong if and only if $(M, \mathcal{N}M)$ is strong.

(d) There are nearness frames $(L, \mathcal{N}L)$ where a dense surjection $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ is not necessarily strict.

We further need the following results as seen in [1].

Lemma 3.1.2. *Let $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ be a uniform frame homomorphism.*

(a) If $a \lhd b$ in L, then $h(a) \lhd h(b)$ in M.

(b) If h is a dense surjection, then $a \lhd b$ in L implies $h_*h(a) \leqslant b$.

(c) If h is a strict surjection, then $x \lhd y$ in M if and only if $h_*(x) \lhd h_*(y)$ in L.

(d) If h is a strict surjection, then for any $a \in L$ and any $x \in M$, we have $a \lhd h_*(x)$ in L if and only if $h(a) \lhd x$ in M.

Definition 3.1.3. (a) A nearness frame $(L, \mathcal{N}L)$ is said to be *complete* if every strict surjection $h : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$ is an isomorphism.

(b) A *completion* of $(L, \mathcal{N}L)$ is a strict surjection $h : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$, where $(M, \mathcal{N}M)$ is a complete nearness frame.

Then, the basic results from the above definitions are (as they appear in [3]).

Lemma 3.1.4. *(a) Every nearness frame has a unique completion.*

(b) The completion of a strong nearness frame is strong.

In this study, a uniform homomorphism $h : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$ between nearness frames will be called *completable* if there exists a uniform $\bar{h} : (CM, C\mathcal{N}M) \to (CL, C\mathcal{N}L)$ (where CM means the completion of M and $C\mathcal{N}M$ is the completion of a nearness on M, and so on) such that the square below commutes.

$$\begin{array}{ccc} (CM, C\mathcal{N}M) & \xrightarrow{\bar{h}} & (CL, C\mathcal{N}L) \\ \downarrow{\scriptstyle \lambda_M} & & \downarrow{\scriptstyle \lambda_L} \\ (M, \mathcal{N}M) & \xrightarrow{h} & (L, \mathcal{N}L) \end{array}$$

Next, we have the following crucial lemma (for details on the proof see [3]).

Lemma 3.1.5. *Any uniform $h : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$ with strong $(M, \mathcal{N}M)$ is completable.*

Given that, as noted in Lemma 3.1.1, a nearness frame $(M, \mathcal{N}M)$ is strong whenever there exists a strict surjection map $h : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$ with strong $(L, \mathcal{N}L)$, we now conclude as in [3], the following result.

Proposition 3.1.6. *Completion is a coreflection for strong nearness frames.*

3.2 Totally strong nearness frames

In this section we discuss the totally strong nearness frames and establish that their category, namely **TStrNFrm**, is coreflective in that of uniformly completely regular nearness frames. The categories for totally strong and uniformly completely regular where introduced in [15].

Definition 3.2.1. Let $(L, \mathcal{N}L)$ be a nearness frame, and $A, B \in \mathcal{N}L$. Write $A \lhd\lhd_s B$ if there is an interpolating sequence of uniform covers (C_{nk}) between A and B, where

$$C_{00} = A, C_{01} = B, C_{nk} = C_{n+1\,2k}, \text{and } C_{nk} \lhd C_{n\,k+1}$$

for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$. In this case we say A *scale refines* B. We call a nearness frame $(L, \mathcal{N}L)$ *totally strong* if every uniform cover A is scale refined by a uniform cover B, and we write **TStrNFrm** for the resulting category.

The next result is instant as it was stated without a proof in [15].

Proposition 3.2.2. *In a nearness frame $(L, \mathcal{N}L)$, $A \lhd\lhd_s B$ implies $A \lhd B$. Hence totally strong implies strong.*

Proof. Given $A \lhd\lhd_s B$, let (C_{nk}) be an interpolating sequence of uniform covers between A and B. Then, by definition, $A = C_{00} \lhd C_{01} = B$. □

Evidently, from the above proposition we have, **TStrNFrm** $\subseteq$ **StrNFrm**. In what follows we need to show that every almost uniform nearness frame is totally strong, so, we need a result which shows that interpolation in the underlying frame L is transferred to it is nearness $\mathcal{N}L$. The next two results together with their proofs appear in [15].

Lemma 3.2.3. *Suppose $(L, \mathcal{N}L)$ is an interpolative nearness frame, and suppose $A, B \in \mathcal{N}L$ with $A \lhd B$. Then there exists $C \in \mathcal{N}L$ such that $A \lhd C \lhd B$.*

Proof. Let $A, B \in \mathcal{N}L$ be such that $A \lhd B$. Then for each $a \in A$, there exists $b_a \in B$ such that $a \lhd b_a$. Since $(L, \mathcal{N}L)$ is interpolative, there is $c_a \in L$ such that $a \lhd c_a \lhd b_a$. Form the set

$$C = \{c_a \in L \mid a \in A\}.$$

Then C is a uniform cover, since A refines it. Moreover $A \lhd C \lhd B$ by the way C is constructed. □

Proposition 3.2.4. *If $(L, \mathcal{N}L)$ is almost uniform, then it is totally strong.*

Proof. Given $(L, \mathcal{N}L)$ an almost uniform nearness frame. Let $B \in \mathcal{N}L$. Since $(L, \mathcal{N}L)$ is strong, there exists $A \in \mathcal{N}L$ such that $A \lhd B$. We use the axiom of countable dependent choice (CDC) (see [12, page 84]) to obtain an interpolative sequence of uniform covers (C_{nk}) between A and B as follows: Put $C_{00} = A$ and $C_{01} = B$. Now, by the above lemma, the nearness $\mathcal{N}L$ has the interpolation property. Suppose C_{ij} has been obtained for all $i < n$ and all $k = 0, 1, \ldots, 2^i$ so that $C_{ik} \lhd C_{ik+1}$. Put

$$C_{n\,2k} = C_{n-1\,k}$$

and, using interpolation in $\mathcal{N}L$, choose a uniform cover $C_{n\,2k+1}$ such that

$$C_{n\,2k} \lhd C_{n\,2k+1} \lhd C_{n\,2(k+1)}.$$

Then (C_{nk}) is the desired sequence, so that $A \lhd\lhd_s B$. Thus $(L, \mathcal{N}L)$ is totally strong. □

The category of almost uniform nearness frames and uniform homomorphisms is denoted by **AUNFrm**. As an observation from the above two results, it should be obvious that if $(L, \mathcal{N}L)$ is a strong nearness frame with the property that whenever $A \lhd B$ in $\mathcal{N}L$, there exists $C \in \mathcal{N}L$ such that $A \lhd C \lhd B$, then $(L, \mathcal{N}L)$ is totally strong.

For use in next set of results we aim to show that the category **TStrNFrm** is closed under completions. In our subsequent discussions we recall from Lemma 3.1.2(a) that if $A \lhd B$ then $h[a] \lhd h[b]$, (i.e $A \lhd B \Rightarrow a \lhd b \Rightarrow h[a] \lhd h[b]$). We shall in a number of instances, use the abbreviation $h[A \lhd B]$ for $h[A] \lhd h[B]$. Consequently the notation $h[A \lhd\lhd_s B]$ shall be an abbreviation for $h[A] \lhd\lhd_s h[B]$. The next result was given without a proof in [15].

Lemma 3.2.5. *Let $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ be a uniform frame map.*

(a) If A scale refines B in $(L, \mathcal{N}L)$, then $h[A]$ scale refines $h[B]$ in $(M, \mathcal{N}M)$.

(b) If (C_{nk}) is a scale of uniform covers of $(L, \mathcal{N}L)$ witnessing $A \lhd\lhd_s B$, then clearly $(h[C_{nk}])$ is a scale of uniform covers of $(M, \mathcal{N}M)$ witnessing $h[A] \lhd\lhd_s h[B]$.

(c) If h is a strict surjection and U scale refines V in $(M, \mathcal{N}M)$, then $h_*[U]$ scale refines $h_*[V]$ in $(L, \mathcal{N}L)$.

(d) *If* (W_{nk}) *is a scale of uniform covers of* $(M, \mathcal{N}M)$ *witnessing* $U \lhd\lhd_s V$*, then, by the strictness of h,* $(h_*[W_{nk}])$ *is a scale of uniform covers of* $(L, \mathcal{N}L)$ *witnessing* $h_*[U] \lhd\lhd_s h_*[V]$.

Proof. For (a). Let $A, B \in \mathcal{N}L$, then $h[A], h[B] \in \mathcal{N}M$ since h is a uniform map. Suppose $A \lhd\lhd_s B$ in $(L, \mathcal{N}L)$. Then $h[A \lhd\lhd_s B] = h[A] \lhd\lhd_s h[B]$ in $(M, \mathcal{N}M)$.
For (b). Suppose (C_{nk}) is a scale of uniform covers of $(L, \mathcal{N}L)$ witnessing $A \lhd\lhd_s B$. Then by (a) above $(h[C_{nk}])$ is a scale of uniform covers of $(M, \mathcal{N}M)$ witnessing $h[A] \lhd\lhd_s h[B]$.
For (c). Suppose h is a strict surjection and $U \lhd\lhd_s V$ in $(M, \mathcal{N}M)$. Then, if $U, V \in \mathcal{N}M$ then $h_*[U], h_*[V] \in \mathcal{N}L$, since h is strict surjective. Thus, $h_*[U \lhd\lhd_s V] = h_*[U] \lhd\lhd_s h_*[V]$ in $(L, \mathcal{N}L)$, by Lemma 3.1.2 (c).
For (d). Suppose (W_{nk}) is a scale of uniform cover of $(M, \mathcal{N}M)$ witnessing $U \lhd\lhd_s V$. Then by (c) above $(h_*[W_{nk}])$ is a scale of uniform covers of $(L, \mathcal{N}L)$ witnessing $h_*[U] \lhd\lhd_s h_*[V]$. □

Further the next result appear in [16] together with it is proof.

Lemma 3.2.6. *Let* $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ *be a strict surjection. Then* $(L, \mathcal{N}L)$ *is totally strong if and only if* $(M, \mathcal{N}M)$ *is totally strong.*

Proof. ($\Rightarrow$) Suppose $(L, \mathcal{N}L)$ is totally strong and let U be a uniform cover of $(M, \mathcal{N}M)$. Then, by strictness, $h_*[U]$ is a uniform cover of $(L, \mathcal{N}L)$. Since $(L, \mathcal{N}L)$ is totally strong, there is a uniform cover A of $(L, \mathcal{N}L)$ that scale refines $h_*[U]$. By what we have observed from Lemma 3.2.5, $h[A]$ is a uniform cover of $(M, \mathcal{N}M)$ scale refining $h[h_*[U]] = U$. Therefore $(M, \mathcal{N}M)$ is totally strong.
($\Leftarrow$) Conversely, suppose $(M, \mathcal{N}M)$ is totally strong and let A be a uniform cover of $(L, \mathcal{N}L)$. By strictness, there is a uniform cover U of M such that $h_*[U] \leqslant A$. Since $(M, \mathcal{N}M)$ is totally strong, there is a uniform cover V of $(M, \mathcal{N}M)$ which scale refines U. Then $h_*[V]$ is a uniform cover of $(L, \mathcal{N}L)$ scale refining $h_*[U]$, and hence scale refining A. Therefore $(L, \mathcal{N}L)$ is totally strong. □

Since completion maps are strict surjections, we give the following result.

Corollary 3.2.7. *A nearness frame is totally strong if and only if its completion is totally strong.*

From the Corollary above we conclude that totally strong nearness frame is closed under completion. Further, we aim to construct the totally strong coreflection of a uniformly completely regular nearness frame.

Definition 3.2.8. (a) By $a \lhd\lhd b$, is meant there is an interpolating sequence (c_{nk}) in L between a and b, where

$$c_{00} = a,\, c_{01} = b,\, c_{nk} = c_{n+1\ 2k}, \text{ and } c_{nk} \lhd c_{n\ k+1}$$

for all $n = 0, 1, \dots$ and $k = 0, 1, \dots, 2^n$.

(b) Let $(L, \mathcal{N}L)$ be a nearness frame and $A, B \in \mathcal{N}L$. We say B *completely refines* A and write $B \lhd\lhd A$ if for any $b \in B$, there exists $a \in A$ such that $b \lhd\lhd a$.

(c) Call a nearness frame $(L, \mathcal{N}L)$ *uniformly completely regular* if every cover $A \in \mathcal{N}L$ is completely refined by a cover $B \in \mathcal{N}L$. (These have been termed "completely strong" nearness frames in [15]).

Proposition 3.2.9. *If $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ is a strict surjection, then for any $a \in L$ and $x \in M$, we have $a \lhd\lhd h_*(x)$ if and only if $h(a) \lhd\lhd x$.*

Proof. This result can be deduced from Lemma 3.1.2 (d). To see this.
($\Rightarrow$) Suppose h is a strict surjection map. Let $a \in L$ and $x \in M$ be such that $a \lhd\lhd h_*(x)$. Then,

$$\begin{aligned} a \lhd\lhd h_*(x) &\Rightarrow h[a \lhd\lhd h_*(x)] \\ &\Rightarrow h(a) \lhd\lhd h[h_*(x)] \\ &\Rightarrow h(a) \lhd\lhd x, \text{ since } h[h_*(x)] = x \text{ by hypothesis.} \end{aligned}$$

($\Leftarrow$) Conversely, suppose $h(a) \lhd\lhd x$. Then,

$$\begin{aligned} h(a) \lhd\lhd x &\Rightarrow h_*[h(a) \lhd\lhd x] \\ &\Rightarrow h_*[h(a)] \lhd\lhd h_*(x) \\ &\Rightarrow a \lhd\lhd h_*(x), \text{ since } h_*[h(a)] = a \text{ by the strictness of } h. \end{aligned}$$

□

The following result appear in [15], without a proof.

Lemma 3.2.10. *In a nearness frame $(L, \mathcal{N}L)$.*

(a) $a \lhd\lhd b$ implies $a \lhd b$.

(b) $a \lhd b$ implies $a \lhd\lhd b$, given that $(L, \mathcal{N}L)$ is interpolative.

Proof. (a). Let $a, b \in L$ and $a \lhd\lhd b$. Then by definition there is an interpolating sequence (c_{nk}) in L between a and b with the property that: $c_{00} = a, c_{01} = b, c_{nk} = c_{n+1\,2k}$ and $c_{nk} \lhd c_{n\,k+1}$ for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$. Then $a = c_{00} \lhd c_{01} = b$. Thus $a \lhd b$.
(b). Suppose $(L, \mathcal{N}L)$ is interpolative and $a \lhd b$ for $a, b \in L$. From CDC, we obtain an interpolating sequence (c_{nk}) between a and b in L, with:

$$c_{00} = a,\ c_{01} = b,\ c_{nk} = c_{n+1\,2k}, \text{ and } c_{nk} \lhd c_{n\,k+1}$$

for $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$. Then (c_{nk}) is the desired interpolating sequence. Thus $a \lhd\lhd b$.

□

From the above lemma, $B \lhd\lhd A$ implies $B \lhd A$ in $\mathcal{N}L$, thus uniformly completely regular nearness frames are strong. Further, we show that the totally strong property is indeed stronger than the uniformly completely regular one. The following lemma and it is proof appear in [15].

Lemma 3.2.11. *Let $(L, \mathcal{N}L)$ be a nearness frame and $A, B \in \mathcal{N}L$. Then $B \lhd\lhd_s A$ implies $B \lhd\lhd A$.*

Proof. Let (C_{nk}) be an interpolating sequence of uniform covers between B and A. In this case

$$B = C_{00}, C_{01} = A, C_{nk} = C_{n+1\,2k} \text{ and } C_{nk} \lhd C_{n\,k+1}$$

for all $n = 0, 1, \ldots$ and all $k = 0, 1, \ldots 2^n$. Let $b \in B$. Since $B \lhd A$, find $a \in A$ such that $b \lhd a$. We show that in fact $b \lhd\lhd a$. For this, using CDC, we obtain an interpolating sequence (c_{nk}) between a and b in L as follows:
Put $c_{00} = b$ and $c_{01} = a$. Pick $c_{ij} \in C_{ij}$ for all $i < n$ and $k = 0, 1, \ldots, 2^i$ such that $c_{ik} \lhd c_{i\,k+1}$. Put

$$c_{n\,2k} = c_{n-1\,k}$$

and find $c_{n\,2k+1} \in C_{n\,2k+1}$ such that

$$c_{n\,2k} \lhd c_{n\,2k+1}.$$

Then (c_{nk}) is the desired interpolating sequence so that $b \lhd\lhd a$. Hence $B \lhd\lhd A$. □

Proposition 3.2.12. *A totally strong nearness frame is uniformly completely regular.*

Proof. Suppose $(L, \mathcal{N}L)$ is totally strong and let $A \in \mathcal{N}L$. So we have $B \in \mathcal{N}L$ such that $B \lhd\lhd_s A$. By Lemma 3.2.11, $B \lhd\lhd A$. Hence $(L, \mathcal{N}L)$ is uniformly completely regular.

□

Denote by **UCRNFrm** the category of uniformly completely regular nearness frames. Since **AUNFrm** $\subseteq$ **TStrNFrm** and that **TStrNFrm** $\subseteq$ **UCRNFrm** then the inclusion **AUNFrm** $\subseteq$ **TStrNFrm** $\subseteq$ **UCRNFrm** holds. Thus, almost uniform nearness frames are uniformly completely regular.

Next, we establish that the category of totally strong nearness frames is coreflective in the category of uniformly completely regular nearness frames. First, we construct the coreflection.

Lemma 3.2.13. *Let $(L, \mathcal{N}L)$ be a uniformly completely regular nearness frame. Then $\mathcal{N}L$ is interpolative.*

Proof. Let $A \in \mathcal{N}L$. So there is $B \in \mathcal{N}L$ such that $B \lhd\lhd A$. For each $b \in B$, let $c_b \in L$ and $a_b \in A$ be such that $b \lhd\lhd c_b \lhd\lhd a_b$ (this is because the relation $\lhd\lhd$ is interpolative). Form a set $C = \{c_b \in L \mid b \in B\}$. Then $C \in \mathcal{N}L$ and $B \lhd\lhd C \lhd\lhd A$. In consequence $B \lhd C \lhd A$. Therefore $\mathcal{N}L$ is interpolative. □

Lemma 3.2.14. *Let $(L, \mathcal{N}L)$ be uniformly completely regular. Consider the collection $\underline{\mathcal{N}L} = \{A \in \mathcal{N}L \mid B \lhd\lhd_s A$, for some $B \in \mathcal{N}L\}$. If $x \lhd_{\mathcal{N}L} a$ in L, then $x \lhd_{\underline{\mathcal{N}L}} a$ in L.*

Proof. We need to show that $x \lhd_{\mathcal{N}L} a \Rightarrow x \lhd_{\underline{\mathcal{N}L}} a$. Let $x \lhd_{\mathcal{N}L} a$. Then $\{x^*, a\} \in \mathcal{N}L$. So let $C \in \mathcal{N}L$ be such that $C \lhd\lhd \{x^*, a\}$. Pick $c \in C$. Then $c \lhd\lhd x^*$ or $c \lhd\lhd a$ (so $c \lhd_{\mathcal{N}L} x^*$ or $c \lhd_{\mathcal{N}L} a$). By Lemma 3.2.13, we can build a scale (C_{nk}) of uniform covers in $\mathcal{N}L$ between C and $\{x^*, a\}$. This implies that $\{x^*, a\} \in \underline{\mathcal{N}L}$. Consequently $x \lhd_{\underline{\mathcal{N}L}} a$. □

Lemma 3.2.15. *Let $(L, \mathcal{N}L)$ be a uniformly completely regular nearness frame. Then the collection*

$$\underline{\mathcal{N}L} = \{A \in \mathcal{N}L \mid B \triangleleft\triangleleft_s A, \textit{ for some } B \in \mathcal{N}L\},$$

is a totally strong nearness on L.

Proof. First we need to show that $\underline{\mathcal{N}L}$ is a nearness. Let $A, B \in \underline{\mathcal{N}L}$, then we have $C, D \in \mathcal{N}L$ such that $C \triangleleft\triangleleft_s A$ in $\mathcal{N}L$ and $D \triangleleft\triangleleft_s B$ in $\mathcal{N}L$. Since $\triangleleft\triangleleft_s$ is closed under finite meets, then $C \wedge D \triangleleft\triangleleft_s A \wedge B$ in $\mathcal{N}L$, so $A \wedge B \in \underline{\mathcal{N}L}$. If $A \leqslant C \in \mathrm{Cov}(L)$ with $A \in \underline{\mathcal{N}L}$, then $C \in \mathcal{N}L$, since $\mathcal{N}L$ is a nearness and $A \in \mathcal{N}L$. Thus, $C \in \underline{\mathcal{N}L}$, since $B \triangleleft\triangleleft_s A \leqslant C$ in $\mathcal{N}L$ implies $B \triangleleft\triangleleft_s C$ in $\mathcal{N}L$. Therefore $\underline{\mathcal{N}L}$ is a filter relative to refinement ordering.
For admissibility, let $a, x \in L$. Then, since from Lemma 3.2.14 we have that $x \triangleleft_{\mathcal{N}L} a \Rightarrow x \triangleleft_{\underline{\mathcal{N}L}} a$, we see that

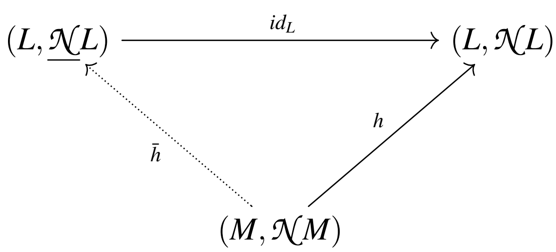
$$a = \bigvee\{x \in L \mid x \triangleleft_{\mathcal{N}L} a\} \leqslant \bigvee\{x \in L \mid x \triangleleft_{\underline{\mathcal{N}L}} a\} \leqslant a.$$

Thus $\underline{\mathcal{N}L}$ is admissible.
Next we show that $\underline{\mathcal{N}L}$ is totally strong. Let $A \in \underline{\mathcal{N}L}$. Then there exists $B \in \mathcal{N}L$ such that $B \triangleleft\triangleleft_s A$ in $\mathcal{N}L$. Since $\mathcal{N}L$ is interpolative by Lemma 3.2.13, let $C \in \mathcal{N}L$ be such that $B \triangleleft\triangleleft_s C \triangleleft\triangleleft_s A$ in $\mathcal{N}L$. Then $C \in \underline{\mathcal{N}L}$. Thus $C \triangleleft\triangleleft_s A$ in $\underline{\mathcal{N}L}$. Therefore $\underline{\mathcal{N}L}$ is totally strong.

□

Proposition 3.2.16. **TstrNFrm** *is a coreflective subcategory of* **UCRNFrm**. *In particular, if $(L, \mathcal{N}L)$ is a uniformly completely regular nearness frame, then $(L, \underline{\mathcal{N}L})$ is its totally strong coreflection with the identity map id_L being the coreflection arrow.*

Proof. Let $h : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$ be a uniform frame homomorphism with $(M, \mathcal{N}M)$ totally strong and $(L, \mathcal{N}L)$ a uniformly completely regular. We need to show that there exists a unique uniform homomorphism $\bar{h} : (M, \mathcal{N}M) \to (L, \underline{\mathcal{N}L})$ making the triangle below commutes, i.e. $id_L \circ \bar{h} = h$



We define $\bar{h}$ by $\bar{h}(x) = h(x)$. Then $\bar{h}$ is a frame homomorphism. To see that $\bar{h}$ is uniform, let $D \in \mathcal{N}M$. Since h is a uniform homomorphism, $h[D] \in \mathcal{N}L$. Since $\mathcal{N}M$ is totally strong, there is $\acute{D} \in \mathcal{N}M$ that scale refines D in $\mathcal{N}M$ (i.e $\acute{D} \lhd\lhd_s D$) and $h[\acute{D}] \lhd\lhd_s h[D]$ in $\mathcal{N}L$ by Lemma 3.2.5. Since every totally strong nearness frame is strong, we have $h[\acute{D}] \lhd_{\mathcal{N}L} h[D]$. Consequently, $h[D] \in \underline{\mathcal{N}L}$. But $\bar{h}[D] = h[D]$. So, $\bar{h}$ is uniform. Clearly, $\bar{h}$ makes the triangle above commute, and since id_L is dense and monic, the uniqueness of $\bar{h}$ follows. $\square$

Chapter 4

Strongness properties in the context of partial frames

4.1 Partial frames

As stated in the abstract we aim to extend our investigations to partial frames. So, in this chapter we begin with a few definitions relating to partial frames. As references for partial frames see [6, 7, 8, 9] and [10].

A *meet-semilattice* L is a poset $(L, \leqslant)$ in which all finite subsets have a meet. A function $f : A \to B$ is a *meet-semilattice map* if it preserves finite meets, as well as the top and the bottom element. *Partial frames* are defined to be meet-semilattices in which certain designated subsets are required to have joins, and finite meets distribute over these joins. We specify the joins for partial frames by means of a selection function as defined below.

Definition 4.1.1. A *selection function* is a rule, usually denoted by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A such that the following conditions hold (for all meet-semilattices A and B):

(S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.

(S2) If $G, H \in \mathcal{S}A$, then $G \wedge H = \{x \wedge y \mid x \in G, y \in H\} \in \mathcal{S}A$.

(S2$'$) If $G, H \in \mathcal{S}A$, then $G \vee H = \{x \vee y \mid x \in G, y \in H\} \in \mathcal{S}A$.

(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x\in G} H_x \in \mathcal{S}A.$$

(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] \mid G \in \mathcal{S}A\} \subseteq \mathcal{S}B:$$

Remark 4.1.2. (a) In case a selection function, $\mathcal{S}$, has been fixed, we shall informally refer to the members of $\mathcal{S}A$ as the *designated subsets* of A.

(b) In a meet-semilattice binary joins may not exist, so it is possible for the designated set given in Axiom (S2$'$) to be empty, which is not a problem.

(c) From Axiom (S4) above, $\mathcal{S}(f[A])$ will be well-defined if $f[A]$ is a meet-semilattice. This is so, because meet-semilattice maps preserve the top element, the bottom element and finite meets.

(d) Another consequence of Axiom (S4) is that, for any meet-semilattice map $f : A \to B$, if $G \in \mathcal{S}A$ then $f[G] \in \mathcal{S}B$.

Partial frames are sometimes called $\mathcal{S}$-frames. We shall sometimes exchangeably write $\mathcal{S}$-frame for a partial frame. Subsequently we define an $\mathcal{S}$-frame.

Definition 4.1.3. Let $\mathcal{S}$ be a selection function.

(a) An *$\mathcal{S}$-frame*, L, is a meet-semilattice that satisfies the following two conditions:

  (i) For all $G \in \mathcal{S}L$, G has a join in L (i.e. $\bigvee G$ exists).

  (ii) For all $x \in L$, for all $G \in \mathcal{S}L$, $x \wedge \bigvee G = \bigvee_{y\in G} x \wedge y$.

(b) Let L and M be $\mathcal{S}$-frames. An *$\mathcal{S}$-frame map* $f : L \to M$ is a meet-semilattice map such that, for all $G \in \mathcal{S}L$, $f(\bigvee G) = \bigvee_{y\in G} f(y)$.

We denote by $\mathcal{S}$**Frm** the category for partial frames and frame homomorphisms.

Remark 4.1.4. (a) From conditions (i) and (ii) in Definition 4.1.3, we note that, since $\{x\} \in \mathcal{S}L$ and for any $G \in \mathcal{S}L$, we have that $\{x\} \wedge G = \{x \wedge y \mid y \in G\} \in \mathcal{S}L$ by Axiom (S2), so $\bigvee_{y\in G} x \wedge y$ exists.

(b) By condition (b) in Definition 4.1.3, we can clearly see that for $G \in \mathcal{S}L$, such that $f[G] \in \mathcal{S}M$ by Axiom (S4), then $f(\bigvee G) = \bigvee_{y\in G} f(y)$ exists.

Definition 4.1.5. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame. A subset M of L is called a *sub $\mathcal{S}$-frame* of L if M is an $\mathcal{S}$-frame and the inclusion map $i : M \to L$ is an $\mathcal{S}$-frame map.

The next result appear in [6], and we state without a proof.

Lemma 4.1.6. *For any $\mathcal{S}$-frame L and $M \subseteq L$, the following conditions are equivalent:*

(a) *M is an $\mathcal{S}$-frame and the inclusion map $i : M \to L$ is an $\mathcal{S}$-frame map.*

(b) *M is a sub meet-semilattice of L and $G \in \mathcal{S}M$ implies that $\bigvee_M G = \bigvee_L G$.*

(c) *M satisfies the conditions:*

(i) $0 \in M$ *and* $1 \in M$.

(ii) $x, y \in M$ *implies* $x \wedge y \in M$.

(iii) $G \in \mathcal{S}M$ *implies* $\bigvee_L G \in M$.

We next deliver an applicable notion of cover for an $\mathcal{S}$-frame as it appears in [6], and we state without proof.

Definition 4.1.7. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame.

(a) We call C an *$\mathcal{S}$-cover* of L if $C \in \mathcal{S}L$ and $\bigvee C = 1$.

(b) If C and D are $\mathcal{S}$-covers of L, then $C \wedge D = \{c \wedge d \mid c \in C, d \in D\}$ is an $\mathcal{S}$-cover of L.

(c) If C and D are $\mathcal{S}$-covers of L, we say that C *refines* D and write $C \leqslant D$ if, for all $c \in C$, there exists $d \in D$ such that $c \leqslant d$.

(d) If $a \in L$ and C is an $\mathcal{S}$-cover of L, we set $C_a = \{c \in C \mid c \wedge a \neq 0\}$. Generally, the $\bigvee C_a$ does not always exist, but when it does, we write $Ca = \bigvee C_a$, as usual.

(e) If $a, b \in L$ and C is an $\mathcal{S}$-cover of L, we write $a \lhd_C b$ if $C_a \subseteq \downarrow b$. Here, $\downarrow b = \{t \in L \mid t \leqslant b\}$ as usual. We say that a is *uniformly below* b with respect to C.

(f) If C and D are $\mathcal{S}$-covers of L, we say that C *star-refines* D, and write $C <^* D$, if, for all $c \in C$, there is $d \in D$ such that $c \lhd_C d$.

Lemma 4.1.8. *[6] If $f : L \to M$ is an $\mathcal{S}$-frame map between $\mathcal{S}$-frames and C is an $\mathcal{S}$-cover of L, then $f[C]$ is an $\mathcal{S}$-cover of M.*

Proof. Since f is, in particular, a meet-semilattice map, Axiom (S4) applies, so $C \in \mathcal{S}L$ implies that $f[C] \in \mathcal{S}M$. Since f is an $\mathcal{S}$-frame map, it preserves joins of designated sets, so $f(\bigvee C) = \bigvee f[C]$, giving $\bigvee f[C] = 1$. □

Definition 4.1.9. Let $\mathcal{S}$ be a selection function and L an $\mathcal{S}$-frame. We call $\mathcal{K}L$ an $\mathcal{S}$-*nearness* on L if the following properties are satisfied:

(a) $\mathcal{K}L$ is a non-empty collection of $\mathcal{S}$-covers of L.

(b) For $C, D \in \mathcal{K}L$, $C \wedge D \in \mathcal{K}L$.

(c) If $C \in \mathcal{K}L$ and D is an $\mathcal{S}$-cover of L such that $C \leqslant D$, then $D \in \mathcal{K}L$.

(d) For each $a \in L$, there exists $T \in \mathcal{S}L$ such that $a = \bigvee T$ and, for each $t \in T$, $t \triangleleft_C a$ for some $C \in \mathcal{K}L$.

In what follows we summarize a few definitions on nearness $\mathcal{S}$-frames, omitting some since they are the same as in nearness frames.

Definition 4.1.10. (a) If $\mathcal{K}L$ is an $\mathcal{S}$-nearness on L, we call $(L, \mathcal{K}L)$ a *nearness $\mathcal{S}$-frame*.

(b) A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is *strong* if for all $D \in \mathcal{K}L$ there exists $C \in \mathcal{K}L$ such that for all $c \in C$ there exists $d \in D$ and $E \in \mathcal{K}L$ such that $c \triangleleft_E d$. In this case, we write $C \triangleleft D$.

(c) A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is a *uniform $\mathcal{S}$-frame* if for all $D \in \mathcal{K}L$ there exists $C \in \mathcal{K}L$ with $C <^* D$. We then call $\mathcal{K}L$ an $\mathcal{S}$-*uniformity* on L.

(d) A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is *totally strong* if every uniform $\mathcal{S}$-cover $A \in \mathcal{K}L$ is scale refined by a uniform $\mathcal{S}$-cover $B \in \mathcal{K}L$. Written $B \triangleleft\triangleleft_s A$ (definitions for uniform $\mathcal{S}$-cover and scale refinement are equivalent to those given in Chapter 3).

(e) A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is *almost uniform* if it is strong and the relation $\triangleleft$ interpolates.

(f) A nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ is *uniformly completely regular* if every $\mathcal{S}$-cover $A \in \mathcal{K}L$ is completely refined by an $\mathcal{S}$-cover $B \in \mathcal{K}L$. Written $B \lhd\lhd A$.

(g) Let $(L, \mathcal{K}L)$, $(M, \mathcal{K}M)$ be nearness $\mathcal{S}$-frames. Then an $\mathcal{S}$-frame map $f : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ is said to be *uniform* if $f : L \to M$ is an $\mathcal{S}$-frame map and, for each $C \in \mathcal{K}L$, $f[C] \in \mathcal{K}M$.

We write **N$\mathcal{S}$Frm** for the category of nearness $\mathcal{S}$-frames and uniform frame homomorphisms. The following result is given in [6] without a proof.

Lemma 4.1.11. *Every uniform $\mathcal{S}$-frame is strong.*

Proof. Suppose $(L, \mathcal{K}L)$ is a uniform $\mathcal{S}$-frame, then for all $B \in \mathcal{K}L$ there is $C \in \mathcal{K}L$ such that $B <^* C$ in $\mathcal{S}$ - uniformity $\mathcal{K}L$. Then, by definition for all $b \in B$, there exists $c \in C$ such that $b \lhd_B c$. Thus $B \lhd C$ in $(L, \mathcal{K}L)$. Thus $(L, \mathcal{K}L)$ is strong. □

We impose another axiom on all selection functions, to be used in the result below and in Proposition 4.1.17.

(S5): For any $\mathcal{S}$-frame L, if M is a sub $\mathcal{S}$-frame of L, $G \subseteq M$ and G is a designated subset of L, then G is a designated subset of M.

We state the next result together with it's proof as seen in [6].

Lemma 4.1.12. *For any $\mathcal{S}$-frame L and $J \subseteq L$ define*

$$\langle J \rangle = \{x \in L \mid x = \bigvee H_x \text{ for some } H_x \in \mathcal{S}L\} \cup \{0\}$$

where any such H_x consists of finite meets of elements of J. Then,

(a) $\langle J \rangle$ is a sub $\mathcal{S}$-frame of L.

(b) If J is a sub $\mathcal{S}$-frame of L, then $\langle J \rangle = J$.

Proof. (a) $0 \in \langle J \rangle$ by definition. $1 \in \langle J \rangle$ since $\{1\} \in \mathcal{S}L$ and 1 is the meet of the empty set. Suppose that $x, y \in \langle J \rangle$. We need only consider the case where $x \neq 0$, $y \neq 0$. Take $H_x, H_y \in \mathcal{S}L$ such that all their elements are finite meets of members of J and $x = \bigvee H_x$, $y = \bigvee H_y$. Then Axiom (S2) guarantees that $\{a \wedge b \mid a \in H_x, b \in H_y\} \in \mathcal{S}L$ and clearly all elements of this set are finite meets of members of J. Then $x \wedge y = \bigvee \{a \wedge b \mid a \in H_x, b \in H_y\}$. So $x \wedge y \in \langle J \rangle$. So far, it has been established that

$\langle J \rangle$ is a sub meet-semilattice of L. Take $G \in \mathcal{S}\langle J \rangle$. If $G = \emptyset$, then $\bigvee G = 0 \in \langle J \rangle$. Otherwise, for each $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}L$ such that all elements of H_x are finite meets of elements of J. Since $\langle J \rangle$ is a sub meet-semilattice of L, $G \in \mathcal{S}L$. Apply Axiom (S3) to get $\bigcup_{x \in G} H_x \in \mathcal{S}L$. Then $\bigvee(\bigcup_{x \in G} H_x) \in \langle J \rangle$. From Axiom (S3), $G = \bigcup_{x \in G} H_x$. Thus $\bigvee G = \bigvee(\bigcup_{x \in G} H_x) \in \langle J \rangle$, so $\bigvee G \in \langle J \rangle$, as required.
(b) Suppose J is a sub $\mathcal{S}$-frame of L. Let $x \in J \subseteq L$ (i.e $x \in L$). Then $x = \bigvee H_x$ for some $H_x \in \mathcal{S}L$. Thus $x \in \langle J \rangle$. Hence $J \subseteq \langle J \rangle$. For the converse, one uses Axiom (S5). □

Definition 4.1.13. We refer to the notation $\langle J \rangle$ used in Lemma 4.1.12, as the *sub $\mathcal{S}$-frame of L generated by J.*

Remark 4.1.14. We note that our definition of selection function (see Definition 4.1.1) requires only the non-empty set to be selected. But if we violate that condition (by selecting the empty set), the definition of $\langle J \rangle$ would automatically result in $0 \in \langle J \rangle$. So, our best choice is to keep the number of axioms to a minimum.

Next we state the result in [6], which allows us to construct uniform images for nearness $\mathcal{S}$-frames using Axiom (S1) to Axiom (S5).

Proposition 4.1.15. *Suppose that L and M are $\mathcal{S}$-frames, $(L, \mathcal{K}L)$ is a nearness $\mathcal{S}$-frame and $f : L \to M$ is an $\mathcal{S}$-frame map. Define $f(L, \mathcal{K}L) = (f[L], \langle f[\mathcal{K}L] \rangle)$, where $f[L] = \{f(x) | x \in L\}$ and $\langle f[\mathcal{K}L] \rangle = \{D \in \mathcal{S}M | D > f[C]$ for some $C \in \mathcal{K}L\}$. Then $f(L, \mathcal{K}L)$ is a nearness $\mathcal{S}$-frame, and $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ is a uniform map.*

Proof. First we show that $f[L]$ is an $\mathcal{S}$-frame. As stated in Remark 4.1.2 (c), $f[L]$ is a meet-semilattice. Now take $H \in \mathcal{S}(f[L])$. By Axiom (S4), $H = f[G]$, for some $G \in \mathcal{S}L$. Since f is an $\mathcal{S}$-frame map, $f(\bigvee G) = \bigvee f[G] = \bigvee H$, so $\bigvee H$ does exist. To check the required distributivity, take $y \in f[L]$. Then $y = f(x)$ for some $x \in L$. Then

$$y \wedge \bigvee H = f(x) \wedge \bigvee f[G] = f(x) \wedge f(\bigvee G) = f(x \wedge \bigvee G) = f(\bigvee_{a \in G} x \wedge a)$$
$$= \bigvee_{a \in G} f(x \wedge a) = \bigvee_{a \in G} f(x) \wedge f(a) = \bigvee_{b \in H} y \wedge b.$$

We note that the fifth equality uses the fact that $\{x \wedge a \mid a \in G\} \in \mathcal{S}L$ which follows from the facts that $\{x\}, G \in \mathcal{S}L$ and Axiom (S2).
Next we show that $\langle f[\mathcal{K}L] \rangle$ is an $\mathcal{S}$- frame on $f[L]$. By Lemma 4.1.8, if $C \in \mathcal{K}L$, then $f[C]$ is an $\mathcal{S}$-cover of M, and also of $f[L]$, by (S4). So $\langle f[\mathcal{K}L] \rangle$ does consist of $\mathcal{S}$-covers

of $f[L]$. If $D_1, D_2 \in \langle f[\mathcal{K}L]\rangle$, then $D_1 \geq f[C_1]$ and $D_2 \geq f[C_2]$ for some $C_1, C_2 \in \mathcal{K}L$. Then $D_1 \wedge D_2 \geq f[C_1] \wedge f[C_2] = f[C_1 \wedge C_2]$ and $C_1 \wedge C_2 \in \mathcal{K}L$. Further, $D_1 \wedge D_2 \in \mathcal{S}M$, by Axiom (S2). So $\langle f[\mathcal{K}L]\rangle$ is closed under finite meets.

It is clear that, if $D \in \langle f[\mathcal{K}L]\rangle$ and E is an $\mathcal{S}$-cover of $f[L]$ such that $E \geq D$, then $E \in \langle f[\mathcal{K}L]\rangle$. For the compatibility condition, begin with $b \in f[L]$. Then $b = f(a)$ for some $a \in L$. Now $a = \bigvee T$ for some $T \in \mathcal{S}L$ such that $t \in T$ implies that $t \lhd_C a$ for some $C \in \mathcal{K}L$. Then $f[C] \in \langle f[\mathcal{K}L]\rangle$ and $f(a) = f(\bigvee T) = \bigvee f[T]$, since f is an $\mathcal{S}$-frame map. We conclude the proof by showing that $t \lhd_C a$ implies that $f(t) \lhd_{f[C]} f(a)$:

$f[C]_{f(t)} = \{f(c) \mid c \in C, f(c) \wedge f(t) \neq 0\} \subseteq \{f(c) \mid c \in C, c \wedge t \neq 0\} = f[C_t] \subseteq f[\downarrow a] \subseteq \downarrow f(a)$.

The fact that $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ is then uniform is clear, since $C \in \mathcal{K}L$ implies that $f[C] \in \langle f[\mathcal{K}L]\rangle$, as remarked above. □

Definition 4.1.16. Let $(L, \mathcal{K}L)$ and $(M, \mathcal{K}M)$ be nearness $\mathcal{S}$- frames. We call $(L, \mathcal{K}L)$ a *sub nearness $\mathcal{S}$- frame* of $(M, \mathcal{K}M)$ if L is a sub $\mathcal{S}$-frame of M and $\mathcal{K}L \subseteq \mathcal{K}M$. We note that this is equivalent to the inclusion $i : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ being a uniform map. We then write $(L, \mathcal{K}L) \leqslant (M, \mathcal{K}M)$, where $\leqslant$ is a partial order between the two nearness.

We state the proof of the following proposition, as it appears in [6].

Proposition 4.1.17. *Let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame. The collection of all sub nearness $\mathcal{S}$-frames of $(L, \mathcal{K}L)$ forms a complete lattice.*

Proof. The relation $\leqslant$ given in Definition 4.1.16 is indeed a partial order. The bottom element is clearly the two element frame with its unique $\mathcal{S}$-nearness (except in the case where L is degenerate, in which case it is L itself). Let $\{L_\alpha, \mathcal{K}L_\alpha \mid \alpha \in I\}$ be a non-empty collection of sub nearness $\mathcal{S}$-frames of $(L, \mathcal{K}L)$. Let $\tilde{L}$ be the sub $\mathcal{S}$-frame of L generated by $\bigcup_{\alpha \in I} L_\alpha$.

Define $\mathcal{K}\tilde{L}$ as follows: $C \in \mathcal{K}\tilde{L}$ if and only if $C \in \mathcal{S}\tilde{L}$ and there exists a natural number n and $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$ for $j = 1, \dots, n$ such that $D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_n} \leqslant C$.

We now show that $(\tilde{L}, \mathcal{K}\tilde{L})$ is the join of $\{L_\alpha, \mathcal{K}L_\alpha \mid \alpha \in I\}$, by noting the following points:

(a) For each $\alpha \in I$, $\mathcal{K}L_\alpha \subseteq \mathcal{K}\tilde{L}$.

(b) For $a, b \in L_\alpha$, $a \lhd b$ in $(L_\alpha, \mathcal{K}L_\alpha)$ implies that $a \lhd b$ in $(\tilde{L}, \mathcal{K}\tilde{L})$, since $C_a \subseteq \downarrow b$ for some $C \in \mathcal{K}L_\alpha$ gives $C_a \subseteq \downarrow b$ for that same $C \in \mathcal{K}\tilde{L}$.

(c) $\mathcal{K}\tilde{L}$ is closed under finite meets.

(d) If $C \in \mathcal{K}\tilde{L}$, $D \in \mathcal{S}\tilde{L}$ and $C \leqslant D$, then $D \in \mathcal{K}\tilde{L}$.

(e) Take $a \in \tilde{L}$. The case $a = 0$ presents no difficulties. Write $a = \bigvee H$ for some $H \in \mathcal{S}L$ such that all the elements of H are finite meets of elements of $B = \bigcup_{\alpha \in L} L_\alpha$. Fix $x \in H$. Write $x = b_{\alpha_1} \wedge \cdots \wedge b_{\alpha_n}$ for some $b_{\alpha_j} \in L_{\alpha_j}$. For each $j = 1, \ldots, n$, $b_{\alpha_j} = \bigvee G_{\alpha_j}$ for some $G_{\alpha_j} \in \mathcal{S}L_{\alpha_j}$ such that $u \in G_{\alpha_j} \Rightarrow u \lhd b_{\alpha_j}$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$, and hence in $(\tilde{L}, \mathcal{K}\tilde{L})$.
Let $Z_x = G_{\alpha_1} \wedge \cdots \wedge G_{\alpha_n}$. Then $Z_x \in \mathcal{S}\tilde{L}$ and $w \in Z_x \Rightarrow w \lhd x$ in $(\tilde{L}, \mathcal{K}\tilde{L})$. Further, $x = \bigvee Z_x$. Now let $Z = \bigcup_{x \in H} Z_x$. Then $a = \bigvee Z$ and $w \in Z \Rightarrow w \lhd a$ in $(\tilde{L}, \mathcal{K}\tilde{L})$. What remains to be shown is that $Z \in \mathcal{S}\tilde{L}$. Since $a = \bigvee H$, $H \in \mathcal{S}L$ and for all $x \in H$, $x = \bigvee Z_x$ for $Z_x \in \mathcal{S}L$, Axiom (S3) guarantees that $Z \in \mathcal{S}L$. Since $\tilde{L}$ is a sub $\mathcal{S}$-frame on L, $Z \subseteq \tilde{L}$ and $Z \in \mathcal{S}L$, Axiom (S5) guarantees that $Z \in \mathcal{S}\tilde{L}$.

(f) $(\tilde{L}, \mathcal{K}\tilde{L})$ is a nearness $\mathcal{S}$-frame from (c), (d) and (e) above.

(g) $(\tilde{L}, \mathcal{K}\tilde{L})$ is a sub nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$, since $\tilde{L}$ is a sub $\mathcal{S}$-frame of L and $\mathcal{K}\tilde{L} \subseteq \mathcal{K}L$. The latter follows since $\mathcal{K}L_\alpha \subseteq \mathcal{K}L$ and if $C \in \mathcal{S}\tilde{L}$ with $D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_n} \leqslant C$, for some $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$, then $D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_n} \in \mathcal{K}L$ and so $C \in \mathcal{K}L$.

(h) If $(M, \mathcal{K}M)$ is a sub nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$ such that $(L_\alpha, \mathcal{K}L_\alpha) \leqslant (M, \mathcal{K}M)$ for all $\alpha \in I$, then L_α is a sub $\mathcal{S}$-frame of M and $\mathcal{K}L_\alpha \subseteq \mathcal{K}M$ for all $\alpha \in I$. So, $\tilde{L}$ is a sub $\mathcal{S}$-frame of M and $\mathcal{K}\tilde{L} \subseteq \mathcal{K}L$. We see that $(\tilde{L}, \mathcal{K}\tilde{L})$ is indeed the join of $\{(L_\alpha, \mathcal{K}L_\alpha) \mid \alpha \in I\}$, as required.

□

We shall denote by P an arbitrary property that a nearness $\mathcal{S}$-frame might have. As in [6], we introduce the idea of a P-approximation of a nearness $\mathcal{S}$-frame and use it to construct a functor from **N$\mathcal{S}$Frm** to itself, by assuming that the property P is preserved by uniform images.

Definition 4.1.18. Let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame.

(a) We call those sub nearness $\mathcal{S}$-frames of $(L, \mathcal{K}L)$ that have property P the *P-approximations* of $(L, \mathcal{K}L)$.

(b) Define $\Gamma_P(L, \mathcal{K}L)$ to be the join of all the P-approximations of $(L, \mathcal{K}L)$ (as given in Proposition 4.1.17).

By Proposition 4.1.17, $\Gamma_P(L, \mathcal{K}L)$ is a nearness $\mathcal{S}$-frame. As in [6] we make no claim that $\Gamma_P(L, \mathcal{K}L)$ necessarily satisfies property P, but we will be mostly interested in those properties P where it does.
The $\Gamma_P(L, \mathcal{K}L)$ which is defined in a given nearness $\mathcal{S}$-frame that has no P-approximations is referred to an *empty join.*

Definition 4.1.19. Let $h : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ be a uniform map between nearness $\mathcal{S}$-frames.

(a) We define $h(L, \mathcal{K}L) = (h[L], h[\mathcal{K}L])$ where $h[L] = \{h(x) \mid x \in L\}$ and $h[\mathcal{K}L] = \{h[C] \mid C \in \mathcal{K}L\}$. It is straightforward to see that $h(L, \mathcal{K}L)$ is a sub nearness $\mathcal{S}$- frame of $(M, \mathcal{K}M)$ and that $h : (L, \mathcal{K}L) \to h(L, \mathcal{K}L)$ is a uniform map.

(b) If a property P satisfies the condition that, whenever a nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ has property P, then $h(L, \mathcal{K}L)$ has property P for any uniform h, we say that P is *preserved by uniform images*.

Further, we present the following result as in [6].

Proposition 4.1.20. *Let P be a property that is preserved by uniform images. Then* $\Gamma_P : \mathbf{N\mathcal{S}Frm} \to \mathbf{N\mathcal{S}Frm}$ *is a functor.*

Proof. Γ_P was defined on objects in Definition 4.1.18. We define Γ_P on morphisms as follows. Let $h : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ be a uniform map between nearness $\mathcal{S}$-frames. We need to show that $\Gamma_P h : \Gamma_P(L, \mathcal{K}L) \to \Gamma_P(M, \mathcal{K}M)$ given by restricting the domain and codomain of h is again a uniform map. For briefness we write,

$$\Gamma_P(L, \mathcal{K}L) = (\tilde{L}, \mathcal{K}\tilde{L}) \text{ and } \Gamma_P(M, \mathcal{K}M) = (\tilde{M}, \mathcal{K}\tilde{M}).$$

Let $\{(L_\alpha, \mathcal{K}L_\alpha) \mid \alpha \in L\}$ be the collection of all P- approximations of $(L, \mathcal{K}L)$. Then by assumption, $h(L_\alpha, \mathcal{K}L_\alpha)$ is a P- approximation of $(M, \mathcal{K}M)$ for all $\alpha \in I$. So, $h(L_\alpha, \mathcal{K}L_\alpha) \leqslant (\tilde{M}, \mathcal{K}\tilde{M})$. Then $h[L_\alpha] \subseteq \tilde{M}$ for all $\alpha \in I$, giving $h[\tilde{L}] \subseteq \tilde{M}$. Additionally, $h[\mathcal{K}L_\alpha] \subseteq \mathcal{K}\tilde{M}$ for all $\alpha \in I$, so $\langle h[\mathcal{K}\tilde{L}]\rangle \subseteq \mathcal{K}\tilde{M}$.
Γ_P preserves identities since $\Gamma_P(a) = 0$ iff $a = 0$ and $\Gamma_P(b) = 1$ iff $b = 1$. For the

composition Let $g : (\underline{L}, \mathcal{K}\underline{L}) \to (\underline{M}, \mathcal{K}\underline{M})$ be another uniform map between nearness $\mathcal{S}$-frames . Then $\Gamma_P(hg) = (\Gamma_P h)(\Gamma_P g)$, since the composition of uniform maps is also uniform and that the restrictions are also uniform. □

Next we give the construction of a coreflection for nearness $\mathcal{S}$-frames as it appears in [6].

Theorem 4.1.21. *Let P be a property satisfying the conditions:*

(a) P is preserved by uniform images, and

(b) for any nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$, the join $\Gamma_P(L, \mathcal{K}L)$ of all P-approximations of $(L, \mathcal{K}L)$ has property P.

Then the nearness $\mathcal{S}$-frames with property P form a full monocoreflective subcategory of all nearness $\mathcal{S}$-frames.

Proof. Let P be a property described as above and let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame. We show that the inclusion map $\eta_P : \Gamma_P(L, \mathcal{K}L) \to (L, \mathcal{K}L)$ is the desired coreflection map. Let $(M, \mathcal{K}M)$ be a nearness $\mathcal{S}$-frame with property P and
$f : (M, \mathcal{K}M) \to (L, \mathcal{K}L)$ is a uniform map. Now $f(M, \mathcal{K}M)$ is a sub nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$ and has property P, so is a P-approximation of $(L, \mathcal{K}L)$. This makes the inclusion $i : f(M, \mathcal{K}M) \to \Gamma_P(L, \mathcal{K}L)$ a uniform map and we have the following commuting diagram:

$$\begin{array}{ccc} \Gamma_P(L, \mathcal{K}L) & \xrightarrow{\eta_P} & (L, \mathcal{K}L) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle f} \\ f(M, \mathcal{K}M) & \xleftarrow[f]{} & (M, \mathcal{K}M) \end{array}$$

The factorization of f is unique, because η_P is 1 - 1, and hence a monomorphism. □

Definition 4.1.22. We call the coreflection constructed in Theorem 4.1.21 *the P-coreflection of nearness $\mathcal{S}$-frames.*

Further, a morphism $h : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ in **N$\mathcal{S}$Frm** is an *isomorphism* if and only if $h : L \to M$ is an $\mathcal{S}$-frame isomorphism and $\langle h[\mathcal{K}L] \rangle = \mathcal{K}M$.
Moreover, if $f : (L, \mathcal{K}L) \to (M, \mathcal{K}M)$ is a morphism in **N$\mathcal{S}$Frm** and f is 1- 1, then $(L, \mathcal{K}L)$ is *isomorphic* to $f(L, \mathcal{K}L)$ which is a sub nearness $\mathcal{S}$-frame of $(M, \mathcal{K}M)$.

In what follows, we show that every full, isomorphism-closed coreflective subcategory, $\mathbf{H}$, of $\mathbf{N}\mathcal{S}\mathbf{Frm}$ with the coreflection maps that are all 1-1, can be obtained by the same construction given in Theorem 4.1.21. The following result was given without a proof in [6].

Proposition 4.1.23. *Let $\mathbf{H}$ be a full, isomorphism-closed, coreflective sub-category of $\mathbf{N}\mathcal{S}\mathbf{Frm}$ for which the $\mathbf{H}$-coreflection maps are all 1-1. Define P by stating that a nearness $\mathcal{S}$-frame $(L, \mathcal{K}L)$ satisfies P if and only if $(L, \mathcal{K}L)$ is an object of $\mathbf{H}$. Then the P-coreflection and the $\mathbf{H}$-coreflection of any nearness $\mathcal{S}$-frame are isomorphic.*

Proof. Let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame and denote it is $\mathbf{H}$-coreflection map by $\eta : \mathbf{H}(L, \mathcal{K}L) \to (L, \mathcal{K}L)$. As η is 1-1 by hypothesis, then $\eta\mathbf{H}(L, \mathcal{K}L)$ is a sub-nearness $\mathcal{S}$-frame of $(L, \mathcal{K}L)$. Actually, it is a P-approximation of $(L, \mathcal{K}L)$, since it has property P. Thus, $\eta\mathbf{H}(L, \mathcal{K}L) \leqslant \Gamma_P(L, \mathcal{K}L)$.
Let $\{(L_\alpha, \mathcal{K}L_\alpha) \mid \alpha \in I\}$ be the set of all P-approximations of $(L, \mathcal{K}L)$. Let $\alpha \in I$. Then the inclusion map $i : (L_\alpha, \mathcal{K}L_\alpha) \to (L, \mathcal{K}L)$ is a uniform map. Since $(L_\alpha, \mathcal{K}L_\alpha) \in \mathbf{H}$, then i factors through η, this means, there exists a unique uniform map $h : (L_\alpha, \mathcal{K}L_\alpha) \to \mathbf{H}(L, \mathcal{K}L)$ such that $\eta h = i$. Since both η and i are 1-1, then h is also 1-1. Then $h(L_\alpha, \mathcal{K}L_\alpha)$ is a sub-nearness $\mathcal{S}$-frame of $\mathbf{H}(L, \mathcal{K}L)$. Consequently, $\eta h(L_\alpha, \mathcal{K}L_\alpha)$ is a sub-nearness $\mathcal{S}$-frame of $\eta\mathbf{H}(L, \mathcal{K}L)$. So, this makes $(L_\alpha, \mathcal{K}L_\alpha)$ a sub-nearness $\mathcal{S}$-frame of $\eta\mathbf{H}(L, \mathcal{K}L)$ for all $\alpha \in I$. Knowing that $\Gamma_P(L, \mathcal{K}L)$ is the join of all $(L_\alpha, \mathcal{K}L_\alpha)$. Then we have $\Gamma_P(L, \mathcal{K}L) \leqslant \eta\mathbf{H}(L, \mathcal{K}L)$. Hence the equality follows. □

4.2 The strong and totally strong nearness $\mathcal{S}$-frames coreflections

Indeed it is known that frames are a special case of partial frames [8]. In consequence, the main results of Chapter 3 are deducible from those established in this section. However, we have endeavored to present the results in Chapter 3 as such out of pure interest in the display of their technicalities.

It is well known that uniform partial frames and strong nearness partial frames are coreflective in nearness partial frames [6, 7]. So, in this section we are more interested in establishing that totally strong nearness $\mathcal{S}$-frames are coreflective in nearness $\mathcal{S}$-frames, using the techniques used in [6].

Denote by **StrN$\mathcal{S}$Frm** the category of strong nearness $\mathcal{S}$-frames and uniform homomorphisms and by **TStrN$\mathcal{S}$Frm** the subcategory of totally strong nearness $\mathcal{S}$-frames and uniform homomorphisms. We state the next result together with it is proof as it appears in [6].

Proposition 4.2.1. *The strong nearness $\mathcal{S}$-frames form a coreflective subcategory of all nearness $\mathcal{S}$-frames.*

Proof. We apply Theorem 4.1.21.

We start by showing that the uniform image of a strong nearness $\mathcal{S}$-frame is again strong. Let $(L, \mathcal{K}L)$ be strong and $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ provide a uniform image, as defined in Proposition 4.1.15. For any $F \in \langle f[\mathcal{K}L]\rangle$, $F \geq f[E]$ for some $E \in \mathcal{K}L$. Since $\mathcal{K}L$ is strong, there exists $D \in \mathcal{K}L$ such that $D \lhd E$. So, for all $d \in D$, there exists $e \in E$ and $C \in \mathcal{K}L$ such that $d \lhd_C e$, that is, $C_d = \{c \in C : c \wedge d \neq 0\} \subseteq\downarrow e$. Then $f(d) \lhd_{f[C]} f(e)$ as shown in the proof of Proposition 4.1.15. So $f[D] \lhd f[E]$. So we have $f[D] \in \langle f[\mathcal{K}L]\rangle$ such that $f[D] \lhd F$, as required.

For the second condition, let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame, $\{(L_\alpha, \mathcal{K}L_\alpha) \mid \alpha \in I\}$ the set of its strong approximations and $(\tilde{L}, \mathcal{K}\tilde{L})$ their join. (So this is $\Gamma_P(L, \mathcal{K}L)$ where P is the property of being strong.) We show that $(\tilde{L}, \mathcal{K}\tilde{L})$ is strong.

For $C \in \mathcal{K}\tilde{L}$, $C \geq D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_n}$ for some $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$, $j = 1, \ldots, n$. For all j, take $E_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$ such that $E_{\alpha j} \lhd D_{\alpha_j}$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$, and let $E = E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_n}$. Then $E \in \mathcal{K}\tilde{L}$, since $\mathcal{K}L_{\alpha_j} \subseteq \mathcal{K}\tilde{L}$ and $\mathcal{K}\tilde{L}$ is closed under finite meets. We conclude the proof by showing that $E \lhd C$ in $(\tilde{L}, \mathcal{K}\tilde{L})$. For this, begin with $e = e_1 \wedge \cdots \wedge e_n \in E$, where $e_j \in E_{\alpha_j}$. For each $j = 1, \ldots, n$, there exists $d_j \in D_{\alpha_j}$ and $F^j \in \mathcal{K}L_{\alpha_j}$ such that $e_j \lhd_{F^j} d_j$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$. Write $F = F^1 \wedge \cdots \wedge F^n$ and $d = d_1 \wedge \cdots \wedge d_n$.

A straightforward calculation shows that $F_e \subseteq F^j_{e_j} \subseteq\downarrow d_j$ for all $j = 1, \ldots, n$ giving $F_e \subseteq\downarrow d$ and so $e \lhd_F d$ as required. □

Next we provide the promised construction of the totally strong nearness $\mathcal{S}$-frames coreflection in the category of strong nearness $\mathcal{S}$-frames.

Proposition 4.2.2. *The totally strong nearness $\mathcal{S}$-frames form a coreflective subcategory of all strong nearness $\mathcal{S}$-frames.*

Proof. We apply Theorem 4.1.21.
We begin by showing that the uniform image of a totally strong nearness $\mathcal{S}$-frame is again totally strong. Let $(L, \mathcal{K}L)$ be totally strong and $f : (L, \mathcal{K}L) \to f(L, \mathcal{K}L)$ provide a uniform image, as defined in Proposition 4.1.15. For any $F \in \langle f[\mathcal{K}L]\rangle$, $F \geq f[E]$ for some $E \in \mathcal{K}L$. Since $\mathcal{K}L$ is totally strong, there exists $D \in \mathcal{K}L$ such that $D \lhd\!\lhd_s E$. So for any scale uniform cover (C_{nk}) witnessing $D \lhd\!\lhd_s E$, there is a scale uniform cover $(f(C_{nk}))$ witnessing $f[D] \lhd\!\lhd_s f[E]$. Thus $f[D] \in \langle f[\mathcal{K}L]\rangle$ such that $f[D] \lhd\!\lhd_s F$, as required.
For the second condition, let $(L, \mathcal{K}L)$ be a nearness $\mathcal{S}$-frame, $\{(L_\alpha, \mathcal{K}L_\alpha) \mid \alpha \in I\}$ the set of its totally strong approximations and $(\tilde{L}, \mathcal{K}\tilde{L})$ their join. (So this is $\Gamma_P(L, \mathcal{K}L)$ where P is the property of being totally strong). We show that $(\tilde{L}, \mathcal{K}\tilde{L})$ is totally strong. For $C \in \mathcal{K}\tilde{L}$, $C \geq D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_n}$ for some $D_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$, $j = 1, \ldots, n$. For all j, take $E_{\alpha_j} \in \mathcal{K}L_{\alpha_j}$ such that $E_{\alpha_j} \lhd\!\lhd_s D_{\alpha_j}$ in $(L_{\alpha_j}, \mathcal{K}L_{\alpha_j})$, and let $E = E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_n}$. Then $E \in \mathcal{K}\tilde{L}$, since $\mathcal{K}L_{\alpha_j} \subseteq \mathcal{K}\tilde{L}$ and $\mathcal{K}\tilde{L}$ is closed under finite meets. We conclude the proof by showing that $E \lhd\!\lhd_s C$ in $(\tilde{L}, \mathcal{K}\tilde{L})$. Since $E \lhd\!\lhd_s D_{\alpha_1} \wedge \cdots \wedge D_{\alpha_n} \leqslant C$ and $(L_\alpha, \mathcal{K}L_\alpha) \leqslant (\tilde{L}, \mathcal{K}\tilde{L})$. Then $E \lhd\!\lhd_s C$ in $\mathcal{K}\tilde{L}$ as required. □

Proposition 4.2.3. *The totally strong nearness $\mathcal{S}$-frames form a coreflective subcategory of all nearness $\mathcal{S}$-frames.*

Proof. Since totally strong nearness $\mathcal{S}$-frames are coreflective in strong nearness $\mathcal{S}$-frames by Proposition 4.2.2, and that strong nearness $\mathcal{S}$-frames are coreflective in all nearness $\mathcal{S}$-frames by Proposition 4.2.1. Then totally strong nearness $\mathcal{S}$-frames are coreflective in the category of all nearness $\mathcal{S}$-frames, as required. □

Appendix

Conclusions and recommendations

4.3 Conclusions

In conclusion, there is a relationship between the category of totally strong and uniformly completely regular nearness frames, and thus the category of totally strong nearness frames is coreflective in the category of uniformly completely regular nearness frames. Consequently, the category of totally strong nearness partial frames is coreflective in the category of all nearness partial frames.

4.4 Recommendations

One of the recommendations for future investigation in this area is to verify whether or not the category of uniformly completely regular nearness frames is closed under completions. Another recommendation is to establish whether the categories of uniformly completely regular nearness frames and uniformly completely regular nearness partial frames are coreflective (by means of construction) in nearness frames and in nearness partial frames respectively.

Bibliography

[1] B. Banaschewski, *Uniform Completion in Pointfree Topology,* Topological and algebraic structures in fuzzy sets, Trends Log. Stud. Log. Libr., Kluwer Acad. Publ.,Dordrecht. **20** (2003), 19-56.

[2] B. Banaschewski and A. Pultr, *Cauchy points of uniform and nearness frames,* Quaest. Math. **19** (1996), 101-127.

[3] B. Banaschewski, S. S. Hong and A. Pultr, *On the completion of nearness frames,* Quaest. Math. **21** (1998), 19-37.

[4] C.H. Dowker and D. Papert, *Quotient frames and subspaces,* Proc. London Math. Soc. **16** (3) (1966), 275-296.

[5] T. Dube and M. M. Mugochi, *A note on almost uniform nearness frames,* Quaest. Math. **34** (2011), 247-263.

[6] J. Frith and A. Schauerte, *Uniformities and covering properties for partial frames (I),* Categ. General Alg. Struct. Appl. **2** (1) (2014), 1-27.

[7] J. Frith and A. Schauerte, *Completions of uniform partial frames,* Acta Math. Hungar, **147** (1) (2015), 116-134.

[8] J. Frith and A. Schauerte, *A method for constructing coreflections for nearness frames,* Appl. Categ. Struct. **22** (2014), 5-6.

[9] J. Frith and A. Schauerte, *Coverages give free constructions for partial frames,* Appl. Categ. Struct. **25** (2015), 303-321.

[10] J. Frith and A. Schauerte, *Compactifications of partial frames via strongly regular ideals,* Math. Slovaca, 2016.

[11] P.T. Johnstone, *Stone spaces,* Cambridge studies in Adv. Math. **3**, Camb. Univ. Press, 1983.

[12] P.T. Johnstone, *Notes on logic and set theory,* Cambridge Mathematical Textbooks, Camb. Univ. Press, 1987.

[13] H. Herrlich, *A concept of nearness,* General Topology and its Applications, **5** (1974), 191-212.

[14] S. Mac Lane, *Categories for working mathematician,* Grand. Texts in Math., **5**, Springer-Verlag, Berlin, Heiderlberg New York, 1971.

[15] M. M. Mugochi, *Contributions to the theory of nearness in pointfree topology,* Ph.D. Thesis, Univ. South Africa, 2009.

[16] M. M. Mugochi, *On the smoothness and the totally strong properties for nearness frames,* Int. Sci. Technol. J. Namibia, **1** (1) (2013), 35-46.

[17] I. Naidoo, *Locally finite nearness frames,* MSc. Thesis, Univ. of Durban-Westville, 1998.

[18] J. Picado and A. Pultr, *Frames and locales. Topology without points,* Frontiers in Mathematics, Birkauser/ springer Basel AG, Basel. 2012.