ANALYSIS OF PREDATOR-PREY MODELS WITH INFECTION IN BOTH SPECIES AND HUMAN INTERVENTION.

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Abstract

The intricate interplay between infectious diseases and predator-prey dynamics holds pivotal significance. Within this context, our focus centres on the impact of human intervention through treatment and vaccination. This study delves into a predator-prey system, categorizing the populations as susceptible, infected with the disease, vaccinated when inoculated against infection, and under treatment post-infection. We formulate four comprehensive mathematical models that illustrate varying levels of human intervention: no intervention, intervention in the prey only, intervention in the predator only, as well as intervention in both species simultaneously. Mathematical proofs of model positivity are provided. Following the derivation of equilibrium points, we analyse their stability by examining the signs of the eigenvalues from the Jacobian matrix and using the Routh-Hurwitz criteria. To verify our qualitative analysis findings, we conduct simulations using varied parameters in Matlab. We then draw conclusions regarding the impact that human intervention can have on a predator-prey system with infection. Simulation results indicated that without human intervention, predators faced extinction, whereas with treatment and vaccination in either the prey alone or in both predator and prey, the intervention demonstrated a positive effect, preventing the extinction of any species. Therefore, this study concludes that human intervention plays a crucial role in preventing species extinction.

Keywords: Predator, Prey, Treatment, Vaccination, Equilibrium points, Eigenvalues, Stability analysis, Jacobian matrix, Routh-Hurwitz criteria.

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Dedication

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DECLARATIONS

I, Lutopu Khoa, hereby declare that this study, Analysis of predator-prey models with infection in both species and human intervention, is my own work and is a true reflection of my research, and that this work, or any part thereof has not been submitted for a degree at any other institution.

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Chapter 1

Introduction

In this chapter a brief overview of the topic of the research as well as the aim of the study are given. Following which an outline of the mini-thesis is provided.

1.1 Background of the Study

Species interaction which involves the consumption of one species by another is referred to as a predator-prey system, where the predator eats the prey. The interaction is called predation [9].

An infectious disease can be defined as an illness due to a pathogen or its toxic product, which arises through transmission from an infected person, an infected animal or a contaminated inanimate object to a susceptible host [21].

Kermack and McKendrick subdivided the population into three separate and distinct subgroups. This would happen when one or more infected individual is introduced into a community of susceptible individuals. The disease spreads from the infected to susceptible by contact infection. Each infected person runs through the course of the illness and is finally removed from the number of those who are infected by recovery or death [14]. This is called an SIR model.

The basic reproduction number, is a quantity denoted R_0, that represents the number of secondary infections resulting from a single primary infection into an otherwise susceptible population. It is used to evaluate and assess the disease risk and measure the epidemic risk used as a predictive tool to prevent disease outbreak and to prevent the emergence of an epidemic. R_0 can thus be viewed as a threshold. When $R_0 > 1$, this implies a disease outbreak while $R_0 < 1$ implies that the disease will die out. It is also used in the calculation of the proportion of the population that should be vaccinated, in order to achieve herd immunity [18].

The mini-thesis aims to determine the conditions under which an infection in a predator-prey system will persist or die out. This is done by carefully analysing the equilibrium points for each formulated model. The mini-thesis also aims to determine the effects of treatment and vaccination on the disease's dynamics.

1.2 Problem statement

This study is focused on analysing predator-prey models with infection in both species. Although predation is a natural disease curbing mechanism in the case of infected prey, alone it does not guarantee eradication of the disease without prey extinction, nor that the infection will not spread in the predator population.

1.3 Objectives of the study

The main goal of this research is to study the impact of human intervention on a predator-prey system when infection is present in both species.

The objectives of this study are:

1. Develop a mathematical model to describe the dynamics of a predator-prey system with infection, incorporating the impact of human intervention.
2. Investigate the stability of the equilibrium points in the formulated models, comparing sce-

narios with and without human intervention to understand the system's response to external control measures.

3. Conduct computer simulations to illustrate the analytical and numerical findings, and interpret the biological implications of the findings.

4. Conduct a comparative analysis of the results obtained from models with and without human intervention, providing insights into the efficacy and consequences of human intervention in the context of the predator-prey system with infection.

1.4 Research question

How does the inclusion of infection in both predator and prey populations affect the dynamics of predator-prey models, particularly in the context of disease control and spread considering that predation alone may not ensure disease eradication without prey extinction?

1.5 Outline of the mini-thesis

This mini-thesis is organized as follows: In chapter 1, an introduction to the modelling of predator-prey models with infection is provided. In chapter 2, a literature review of the work that has been done so far regarding infection in a predator-prey model and some human intervention is given. Chapter 3 highlights the preliminary concepts that are used throughout the mini-thesis, these include definitions and stability criteria such as the Routh-Hurwitz formula. In chapter 4, the models are formulated and mathematical analysis is done. Chapter 5 covers the quantitative analysis which includes numerical simulations. In the last chapter, the discussion, recommendations, and conclusion are presented. References and appendix are tabulated thereafter.

Chapter 2

Literature Review

A mathematical model is a representation of a process which takes the form of a set of equations that describe a number of variables [8]. A model is formulated with the aim of describing a mechanism in quantitative terms. Following this, its analysis leads to results that can be tested against the observations. Ideally, the model would also lead to predictions, which, if verified authenticate the formulated model.

The simplest and earliest example of the predator-prey model is the Lotka-Volterra model, which was proposed by Vito Volterra and Alfred James Lotka in the mid 1920s. The model was limited by the fact that it assumed no outside influence and that this species interaction occurred in isolation [17]. As such, it was observed that the populations would fluctuate in a regular cycle due to this lack of significant external variables on the relationship. Following its formulation, a lot of research has been carried out in an effort to develop more realistic models.

Ecology is the scientific study of organisms and their interaction with their environment [2]. Epidemiology is the study of the spread of diseases, in an effort to identify factors that contribute to their occurrence or spread. Mathematical modelling of epidemics in populations, which is an important area of study, involves translating biological assumptions into mathematics [6]. Eco-epidemiology which is a fairly new branch in mathematical biology, deals with both the ecological and epidemiological issues concurrently [19].

Bezabih, Edessa and Rao [3] studied the spread of disease in a predator-prey model with treatment, where the infection is in both species. The disease that is spread through contact. They examined the positivity and boundedness of the solutions and analysed seven equilibrium points using linearisation of the model equations and the Jacobian matrix. They provided the conditions for stability of each equilibrium point. The basic reproduction number was calculated at the disease-free equilibrium point for both the prey and the predator. Numerical simulations were conducted using the DEDiscover software, and it was observed that as treatment increases for the infected prey and predator, there is a decrease in the infected population. Conversely, when treatment decreases, the population also decreases. Providing treatment helps save the population from extinction. It was also observed that an increase in the number of infected prey tends to lower the entire population and the disease can be eradicated from the population through treatment.

Doust, Shirazian, and Shamsabadi [7] formulated a similar model that included the treatment of infected species. They conducted a stability analysis of the model, which involved checking the boundedness, stability of the equilibrium points, and calculating the basic reproduction number R_0 using the next-generation matrix method. A control model with infection was established, and an optimal treatment approach was explored. This control model involved adding control functions designed to treat infected species, with these functions ranging from 0 (indicating no treatment) to 1 (representing full treatment). The control model was examined at the endemic equilibrium point, and conditions for its existence and stability were determined. Numerical simulations were also performed using MATLAB, and the optimal control problem was resolved with the assistance of an iterative method. It was observed that applying the control functions increased the number of susceptibles and significantly reduced the number of infectives.

Similarly, Hugo, Massawe, and Makinde [11] formulated a mathematical model that included infection and treatment in both species. Boundedness and positivity of the system were demonstrated, showing that the system was biologically well-posed. The equilibrium points were calculated and analyzed, first by examining the eigenvalues of the Jacobian matrix at each equilibrium point, and secondly by applying the Routh-Hurwitz criterion. The analysis of the model led to the conclusion that treatment of the infected populations has the effect of reducing the spread of the disease and can prevent the population from going extinct.

Two other important aspects to look at are vaccination and migration in the predator-prey system. Kumar and Kharbanda [15] incorporated these two factors in their study of infection in a predator-prey system. A model was formulated, boundedness of the solutions established and the basic reproduction number of the model is calculated using the next generation matrix method. Stability analysis of the equilibrium points was established using Jacobian matrix of systems. The authors concluded that the inclusion of migration tends to decrease infection, and correspondingly, the impact of migration in reducing the spread of infection is augmented by vaccination in the system.

Cojocaru, Migot, and Jaber [5] proposed a prophylactic treatment strategy for a predator-prey system with an SIS epidemic model spread by contact and predation. They discovered that, overall, prophylactic treatment reduces the infection in both the prey and predator populations. Furthermore, they found that vaccinating the prey alone is an efficient method to reduce infection among the predators while also keeping costs below their maximum levels.

Chapter 3

Preliminary Concepts

In this chapter, essential concepts used throughout the mini-thesis are outlined. It begins with the definitions of some terms that are utilized in dynamical systems and within this mini-thesis. Some of these terms include equilibrium points and the Jacobian matrix. The Routh-Hurwitz criterion, one of the tools used in dynamical systems to assess the stability of equilibrium points, is also explained.

3.1 Dynamical Systems

A dynamical system is one whose state changes with time (t). In applied mathematics, there are two main types of dynamical systems that are encountered. The first one is one in which time is discrete in which case $t \in \mathbb{Z}$ or $t \in \mathbb{N}$. The second is one in which time is continuous in which case $t \in \mathbb{R}$ [1].

Below are the definitions of some key terms that will be used throughout the mini-thesis.

Definition 1 (Equilibrium point)**.** *A state $x_e \in \mathrm{I\!R}^n$ is an equilibrium point of an n-dimensional system $x' = f(x)$ if setting $x_0 = x_e$ implies $x(t) = x_e, \forall t > 0$, hence x_e is an equilibrium point if $f(x_e) = 0$. In other words equilibrium points are the zeros of the vector function $f(x)$.*

The equilibrium points will be analysed to determine their stability. This provides an insight on what to expect from the global behaviour of the system in the long run. This analysis will be conducted by studying the eigenvalues of the Jacobian matrix of the system or via Lyapunov Theory.

Definition 2 (Jacobian matrix)**.** *Let f be the vector function defining a system of n differential equations, then the Jacobian matrix (J) of f is a matrix of functions whose $(ij)^{th}$ entry is given by $\frac{\partial f_i}{\partial x_j}$ as shown below*

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \cdots & \frac{\partial f_3}{\partial x_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{3.1}$$

3.2 Stability Criteria

Analysis of the stability of the equilibrium points will be done as follows:

Theorem 1. *Consider the matrix $J(x_e)$, where all derivatives are evaluated at the equilibrium point x_e,*

a) if all the eigenvalues of $J(x_e)$ have negative real parts, then x_e is a stable equilibrium point.

b) if at least one of the eigenvalues of $J(x_e)$ has a positive real part, then x_e is an unstable equilibrium point.

Definition 3. *A Lyapunov function around a point x^* is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that:*

a) V is continuously differentiable;

b) $V(x) > 0$ *for all* $x \neq x^*$ *and* $V(x^*) = 0$*;*

c) $V'(x) < 0$ *for all* $x \neq x^*$*.*

Theorem 2. *An equilibrium point* x_e *of a system* $x' = f(x)$ *is said to be asymptotically stable if there exists a Lyapunov function around* x_e*.*

The Routh-Hurwitz criterion which is used to check the stability of a dynamical system, is a mathematical tool employed to determine whether all the roots of the polynomial have negative real parts.

Theorem 3 (Routh-Hurwitz Criteria [20])**.** *Consider the polynomial*

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 \tag{3.2}$$

The first two rows of the Routh array are obtained by copying the coefficients of $p(\lambda)$ *as given below*

$$\begin{bmatrix} a_n & a_{n-2} & a_{n-4} & \cdots & \lambda^n \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots & \lambda^{n-1} \\ x_1 & x_2 & x_3 & \cdots & \lambda^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \tag{3.3}$$

Where $x_1 = \dfrac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$, $x_2 = \dfrac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \ldots$
The computation is repeated for subsequent rows until the row labelled λ^0 *is reached.*

The Routh-Hurwitz array is then used to check the stability of a system in the following manner, if all the values in the first column of the array (3.3) have the same sign and there are no sign changes then the system is stable. If there are sign changes, then the number of sign changes equals the number of roots in the right half of the λ*-plane that is the number of roots with positive real parts.*

Descartes' Rule of Signs, originally described by René Descartes, is a method used to gather information about the number of positive real roots of a polynomial. It states that the maximum number of positive roots is equal to the number of sign changes in the sequence of the polynomial's coefficients (excluding zero coefficients), and the difference between these two numbers is always even. Consequently, if there are no sign changes or only one sign change, there will be precisely zero or one positive root, respectively.

Theorem 4 (Descartes's rule of signs [22])**.** *Let* $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ *denote a polynomial with nonzero real coefficients* a_i*, where the* b_i *are integers satisfying* $0 \leq b_0 < b_1 < b_2 < \cdots < b_n$*. Then the number of positive real zeros of* $p(x)$ *(counted with multiplicities) is either equal to the number of variations in sign in the sequence* $a_0, \ldots, a_n$ *of the coefficients or less than that by an even whole number. The number of negative zeros of* $p(x)$ *(counted with multiplicities) is either equal to the number of variations in sign in the sequence of the coefficients of* $p(-x)$ *or less than that by an even whole number.*

The following formula shares similarities with the perfect-square method for quadratic equations and serves as a conventional approach to find a real root of a cubic equation in the form of $ax^3 + bx^2 + cx + d = 0$. Subsequently, the remaining two roots, whether real or complex, can be determined through polynomial division and the quadratic formula.

Theorem 5. *(Solution of cubic polynomials [16]) The cubic polynomial* $P : ax^3 + bx^2 + cx + d = 0$ *has solutions:*

$$x_1 = S + T - \frac{b}{3a};$$

$$x_2 = -\frac{S+T}{2} - \frac{b}{3a} + \frac{i\sqrt{3}}{2}(S - T);$$

$$x_3 = -\frac{S+T}{2} - \frac{b}{3a} - \frac{i\sqrt{3}}{2}(S - T),$$

where:

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}\,, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}},$$

with:

$$Q = \frac{3ac - b^2}{9a^2}\,, \qquad R = \frac{9abc - 7a^2d - 2b^3}{54a^3}.$$

3.3 Positivity Analysis

For a given model to be biologically feasible and well-posed, it is necessary to ensure that all variable values remain non-negative as long as their initial values are non-negative. Alternatively, there must exist a feasible region that is positively invariant, meaning that if the initial values of our variables are within that region, they remain in that region the entire time. The positivity of each of the four models is demonstrated using the same approach as Bhunu, Garira, and Mukandavire [4].

3.4 Boundedness

In biological models, the boundedness of a system serves as a cornerstone, signifying its biological validity and overall well-behaved nature. To establish the biological validity of a model, it is necessary to demonstrate the boundedness of its solutions. This critical step ensures that the variables within the model remain within meaningful and biologically plausible limits, reinforcing the foundation of the model's credibility. By affirming the boundedness of the solutions, we not only validate the biological relevance of the model but also lay the groundwork for robust and meaningful analyses within the intricate framework of eco-epidemiological systems, following a methodology similar to Jawad [12].

Chapter 4

Model Formulation and Qualitative Analysis

In this chapter, the formulation of four predator-prey models with and without human intervention is presented. We begin with a model with no intervention, then proceed to models with intervention in the prey only, intervention in the predator only, and finally, intervention in both species. Subsequently, we provide flow diagrams to visually illustrate the interactions among the species. The equilibrium points of the formulated models are determined and analyzed to check their stability conditions.

4.1 Model without human intervention

4.1.1 Formulation of the model

The model is based on the following assumptions.

1. There is no treatment of infected species or vaccination of susceptible species.
2. The prey grows logistically and in the absence of the prey, the predator will die out.
3. Only the healthy species can reproduce and the disease is not hereditary.

4. The prey population is divided into 2 classes: susceptible prey (S_1) and infected Prey (I_1). Likewise the predator population is divided into 2 classes: susceptible predators (S_2) and infected Predators (I_2).

5. In both species, the susceptible population becomes infected by coming into contact with the infected population.

6. The disease is also spread through the process of predation from infected prey to susceptible predator.

7. There is a natural death rate as well as a disease induced death rate for the infected species.

8. The infected prey is more likely to be caught and infected predators are less efficient at catching prey compared to their healthy counterparts.

The following model can then be derived:

$$
\begin{aligned}
S_1' &= \underbrace{aS_1\left(1-\frac{S_1}{K}\right)}_{\text{growth}}\underbrace{-\beta_1 S_1 I_1}_{\text{infection}}\underbrace{-e_1 S_1 S_2 - e_2 S_1 I_2}_{\text{predation}}\underbrace{-d_1 S_1}_{\text{mortality}} \\
I_1' &= \underbrace{\beta_1 S_1 I_1}_{\text{infection}}\underbrace{-e_3 I_1 S_2 - e_4 I_1 I_2}_{\text{predation}}\underbrace{-(d_1+k_1)I_1}_{\text{mortality}} \\
S_2' &= \underbrace{j_1 S_2(e_1 S_1 + e_3 I_1)}_{\text{predation}}\underbrace{-\beta_2 S_2 I_2}_{\text{infection}}\underbrace{-d_2 S_2}_{\text{mortality}} \\
I_2' &= \underbrace{I_2 j_2(e_2 S_1 + e_4 I_1)}_{\text{predation}}\underbrace{+\beta_2 S_2 I_2}_{\text{infection}}\underbrace{-(d_2+k_2)I_2}_{\text{mortality}}
\end{aligned}
\tag{4.1}
$$

The parameters' description is given below:

Table 4.1: Model 1 description of parameters

Parameter	**Description**
a	Intrinsic growth rate for prey
$\beta_{1,2}$	Infection rate in the prey and predator respectively
e_i	Predation rate with, $e_{susceptible} < e_{infected}$ for the same predator.
$d_{1,2}$	Natural death rate of the prey and predator respectively
$j_{1,2}$	Conversion rate of predation for susceptible and infected predators respectively
K	Carrying capacity of the prey
$k_{1,2}$	Disease induced death rate for prey and predators respectively

The flow chart of the above model is given below in figure 4.1:

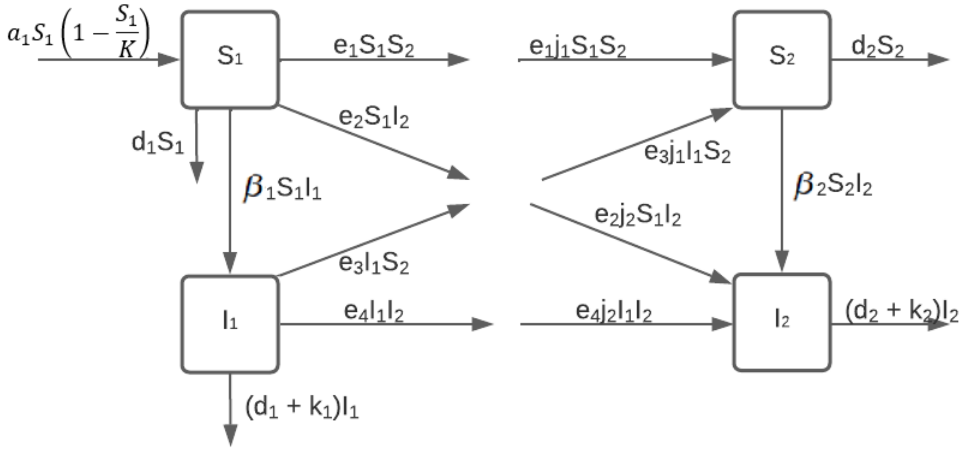

Figure 4.1: Flowchart of Model 1 with no human intervention

4.1.2 Qualitative analysis of the model

4.1.2.1 Positivity of solutions

In this section we show that under the given conditions all the solutions are positive, following a similar proof to that of [4].

Theorem 6 (Positivity)**.** *Let* $S_1(0) \geq 0$ *,* $I_1(0) \geq 0$ *,* $S_2(0) \geq 0$*,* $I_2(0) \geq 0$*. The solutions of (4.1) are non-negative for all* $t \geq 0$*.*

Proof. This will be shown in cases by contradiction.

Case 1. Let t_1 be such that $S_1(t_1) = 0$ and suppose that $S_1^{'}(t_1) < 0$ with $I_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$ for $0 < t < t_1$. In this case substituting these values into the first equation of system (4.1) we get $S_1^{'}(t_1) = 0$ and this is a contradiction to the assumption that $S_1^{'}(t_1) < 0$.

Case 2. Let t_2 be such that $I_1(t_2) = 0$ and suppose that $I_1^{'}(t_2) < 0$ with $S_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$ for $0 < t < t_2$. In this case substituting these values into the second equation of system (4.1) we get $I_1^{'}(t_2) = 0$ and this is a contradiction to the assumption that $I_1^{'}(t_2) < 0$.

Case 3. Let t_3 be such that $S_2(t_3) = 0$ and suppose that $S_2^{'}(t_3) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $I_2(t) > 0$ for $0 < t < t_3$. In this case substituting these values into the third equation of system (4.1) we get $S_2^{'}(t_3) = 0$ and this is contradiction to the assumption that $S_2^{'}(t_3) < 0$.

Case 4. Let t_4 be such that $I_2(t_4) = 0$ and suppose that $I_2^{'}(t_4) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $S_2(t) > 0$ for $0 < t < t_4$. In this case substituting these values into the fourth equation of system (4.1) we get $I_2^{'}(t_4) = 0$ and this is contradiction to the assumption that $I_2^{'}(t_4) < 0$.

Hence all the solutions of (4.1) are non-negative. □

4.1.2.2 Boundedness of solutions

In this section, we demonstrate that under the given conditions, all solutions are bounded above. To establish the boundedness of each population size, it suffices to show that the total population size is bounded. We follow a proof similar to that presented in [12].

Theorem 7 (Boundedness)**.** *Assume that $j_1, j_2 < 1$, then all solutions of the system (4.1) which initiate in $\mathbb{R}^4_+ = \{(S_1, I_1, S_2, I_2),\ S_1 \geq 0,\ I_1 \geq 0,\ S_2 \geq 0,\ I_2 \geq 0\}$ are bounded.*

Proof. Let $N = S_1 + I_1 + S_2 + I_2$ then $\frac{dN}{dt} = S_1' + I_1' + S_2' + I_2'$.

$$\begin{aligned}\frac{dN}{dt} = \;& aS_1\left(1 - \tfrac{S_1}{k}\right) + (j_1 - 1)\,e_1S_1S_2 + (j_2 - 1)e_2S_1I_2 + (j_1 - 1)\,e_3I_1S_2 + (j_2 - 1)e_4I_1I_2 - d_1S_1 - (d_1 + k_1)I_1 \\ & -d_2S_2 - (d_2 + k_2)I_2\end{aligned}$$

since $j_1, j_2 < 1$ we have $(j_1 - 1) < 0$, and $(j_2 - 1) < 0$. Therefore:

$$\begin{aligned}\frac{dN}{dt} &\leq aS_1\left(1 - \tfrac{S_1}{k}\right) - d_1S_1 - (d_1 + k_1)I_1 - d_2S_2 - (d_2 + k_2)I_2 \\ &\leq aS_1 - \tfrac{aS_1^2}{k} - d_1S_1 - (d_1 + k_1)I_1 - d_2S_2 - (d_2 + k_2)I_2 \\ &\leq aS_1 - (d_1S_1 + (d_1 + k_1)I_1 + d_2S_2 + (d_2 + k_2)I_2) \\ &\leq \mu - \eta N\end{aligned}$$

where: $\mu = a\alpha$, $\alpha = \max\{S_1(0), k\}$ and $\eta = \min\{d_1, d_2\}$.

We then have: $\frac{dN}{dt} + \eta N \leq \alpha$ which is a linear first order differential equation. The integration factor is given by $e^{\int \eta dt} = e^{\eta t}$.

$e^{\eta t}\frac{dN}{dt} + \eta e^{\eta t}N \leq \alpha e^{\eta t}$

$\frac{d}{dt}(Ne^{\eta t}) \leq \alpha e^{\eta t}$

$Ne^{\eta t} \leq \frac{\alpha}{\eta}e^{\eta t} + c$

Given the initial condition $N(0) = N_0 \geq 0$, we have $N \leq \frac{\alpha}{\eta} + \left(N_0 + \frac{\alpha}{\eta}\right)e^{-\eta t}$.

As $t \to \infty$, the exponential term $e^{-\eta t} \to 0$. Consequently, we can deduce that $N \leq \frac{\alpha}{\eta}$. Hence all the solutions of (4.1) are bounded and are confined to the region $\Omega = \left\{(S_1, I_1, S_2, I_2) \in \mathbb{R}^4_+ : N \leq \frac{\alpha}{\eta}\right\}$.

□

4.1.2.3 Stability of the equilibrium points

The equilibrium points of our system are obtained by setting the right-hand side of (4.1) to zero. When all the components (S_1, I_1, S_2, I_2) are zero, the equations remain valid, and we obtain the trivial equilibrium point (0,0,0,0). Moreover, for the disease-free equilibrium, we only require I_1^*, and I_2^* to be zero. It therefore follows:
$S_1' = aS_1\left(1 - \frac{S_1}{K}\right) - \beta_1 S_1 I_1 - e_1 S_1 S_2 - e_2 S_1 I_2 - d_1 S_1.$
When $S_1' = 0$ we have:

$$S_1 = 0 \;\; or \;\; S_1 = \frac{K\left(-I_1\beta_1 - I_2 e_2 - S_2 e_1 + a - d_1\right)}{a} \tag{4.2}$$

From the second equation of (4.1),
$I_1' = \beta_1 S_1 I_1 - e_3 I_1 I_2 - e_4 I_1 S_2 - (d_1 + k_1) I_1,$
when $I_1' = 0$ we have:

$$I_1 = 0 \;\; or \;\; -e_3 I_2 + \beta_1 S_1 - e_4 S_2 - d_1 - k_1 = 0 \tag{4.3}$$

We now use the third equation of (4.1),
$S_2' = j_1 S_2 \left(e_3 I_1 + e_1 S_1\right) - \beta_2 S_2 I_2 - d_2 S_2$

When $S_2' = 0$, we have:

$$S_2 = 0 \;\; or \;\; j_1\left(e_3 I_1 + e_1 S_1\right) - \beta_2 I_2 - d_2 = 0 \tag{4.4}$$

$I_2' = j_2 I_2 \left(e_4 I_1 + e_2 S_1\right) + \beta_2 S_2 I_2 - (d_2 + k_2) I_2$

When $I_2' = 0$ we have:

$$I_2 = 0 \;\; or \;\; j_2\left(e_4 I_1 + e_2 S_1\right) + \beta_2 S_2 - \left(d_2 + k_2\right) = 0 \tag{4.5}$$

We have the following equilibrium points:

(i) Trivial equilibrium point: $(S_1, I_1, S_2, I_2) = (0,0,0,0)$

(ii) Infected predator only equilibrium point: $(S_1, I_1, S_2, I_2) = (0,0,0,I_2^*)$. From (4.5), since $I_2 \neq 0$, then we get $d_2 + k_2 = 0$, which is not possible since both d_2 and k_2 are positive. Hence, the equilibrium point is not feasible.

(iii) Healthy predator only equilibrium point: $(S_1, I_1, S_2, I_2) = (0,0,S_2^*,0)$. From (4.4), since $S_2 \neq 0$, we have $d_2 = 0$, which is not possible, and hence the equilibrium point is not feasible.

(iv) Infected prey only equilibrium point: $(S_1, I_1, S_2, I_2) = (0,I_1^*,0,0)$. From (4.3), since $I_1 \neq 0$, we have $d_1 + k_1 = 0$, which is not possible since $d_1, k_1 > 0$, and hence the equilibrium point is not feasible.

(v) Healthy prey only equilibrium point: $(S_1, I_1, S_2, I_2) = (S_1^*,0,0,0)$. From (4.2), we have $S_1^* = K(1 - \frac{d_1}{a})$, and it is feasible only if $a > d_1$.

(vi) Prey-free equilibrium point: $(S_1, I_1, S_2, I_2) = (0,0,S_2^*,I_2^*)$. From (4.4) with $S_2 \neq 0$, we have $I_2^* = -\frac{d_2}{\beta_2}$, and from (4.5), we get $S_2^* = \frac{d_2+k_2}{\beta_2}$. This equilibrium point is not feasible since $I_2^* < 0$.

(vii) Infected prey and predator-only equilibrium point: $(S_1, I_1, S_2, I_2) = (0,I_1^*,0,I_2^*)$. From (4.3) we get $I_2^* = -\frac{d_1+k_1}{e_3}$ from (4.5), and we get $I_1^* = \frac{d_2+k_2}{e_4 j_2}$. This equilibrium point is not feasible since $I_2^* < 0$.

(viii) Infected prey and healthy predator only equilibrium point: $(S_1, I_1, S_2, I_2) = (0,I_1^*,S_2^*,0)$. From

(4.3) we get $I_2^* = -\frac{d_1+k_1}{e_3}$ and from (4.4) we get $I_1^* = \frac{d_2}{e_3 j_1}$. This equilibrium point is not feasible since $I_2^* < 0$.

(ix) Healthy prey and diseased predators equilibrium point: $(S_1, I_1, S_2, I_2) = (S_1^*, 0, 0, I_2^*)$. From (4.2) and (4.5) we get $S_1^* = \frac{d_2+k_2}{e_2 j_2}$ and $I_2^* = \frac{a}{e_2}\left(1 - \frac{d_2+k_2}{K e_2 j_2} - \frac{d_1}{a}\right)$. This equilibrium point is feasible when $1 > \frac{d_1}{a} + \frac{d_2+k_2}{K e_2 j_2}$

(x) Disease-free equilibrium point: $(S_1, I_1, S_2, I_2) = (S_1^*, 0, S_2^*, 0)$. From (4.2) and (4.4) we get $S_1^* = \frac{d_2}{e_1 j_1}$ and $S_2^* = \frac{a}{e_1}\left(1 - \frac{d_2}{j_1 e_1 K} - \frac{d_1}{a}\right)$. This equilibrium point is feasible if $1 > \frac{d_1}{a} + \frac{d_2}{e_1 j_1 K}$.

(xi) Predator-free equilibrium: $(S_1, I_1, S_2, I_2) = (S_1^*, I_1^*, 0, 0)$. From (4.3) and (4.5) we get $S_1^* = \frac{d_1+k_1}{\beta_1}$ and $I_1^* = \frac{K a \beta_1 - K \beta_1 d_1 - a d_1 - a k_1}{{\beta_1}^2 K}$.
This equilibrium point is feasible when $1 > \frac{d_1}{a} + \frac{1}{K\beta_1}(d_1 + k_1)$.

(xii) No healthy prey equilibrium point: $(S_1, I_1, S_2, I_2) = (0, I_1^*, S_2^*, I_2^*)$. This equilibrium point is not feasible since, from (4.5), we get $I_2^* = -\frac{e_4 S_2^* + d_1 + k_1}{e_3}$. Because all the parameters are positive and $S_2^* > 0$, we have $I_2^* < 0$.

(xiii) Infection in the predator only equilibrium point: $(S_1, I_1, S_2, I_2) = (S_1^*, 0, S_2^*, I_2^*)$ where

$$\begin{aligned} S_1^* &= \frac{K(a\beta_2 - d_1\beta_2 - d_2 e_1 + d_2 e_2 - e_1 k_2)}{K e_1 e_2 j_1 - K e_1 e_2 j_2 + a\beta_2} \\ S_2^* &= \frac{d_2 + k_2 - e_2 j_2 S_1^*}{\beta_2} \\ I_2^* &= \frac{e_1 j_1 S_1^* - d_2}{\beta_2} \end{aligned}$$

This equilibrium point is feasible if $a\beta_2 + d_2 e_2 > d_1\beta_2 + d_2 e_1 + e_1 k_2$ and $\frac{d_1}{e_1 j_1} < S_1^* < \frac{d_2+k_2}{e_2 j_2}$.

(xiv) No healthy predators equilibrium point: $(S_1, I_1, S_2, I_2) = (S_1^*, I_1^*, 0, I_2^*)$ where

$$\begin{aligned} S_1^* &= \frac{K(a e_4 j_2 + d_1 e_2 j_2 - d_1 e_4 j_2 + e_2 j_2 k_1 - \beta_1 d_2 - \beta_1 k_2)}{a e_4 j_2} \\ I_1^* &= \frac{-S_1^* e_2 j_2 + d_2 + k_2}{j_2 e_4} \\ I_2^* &= \frac{S_1^* \beta_1 - d_1 - k_1}{e_4} \end{aligned}$$

This equilibrium point is feasible if $a e_4 j_2 + d_1 e_2 j_2 + e_2 j_2 k_1 > \beta_1 k_2 + \beta_1 d_2 + d_1 e_4 j_2$ and $\frac{d_1+k_1}{\beta_1} < S_1^* < \frac{d_2+k_2}{e_2 j_2}$.

(xv) No disease in the predator equilibrium point: $(S_1,I_1,S_2,I_2) = (S_1^*,I_1^*,S_2^*,0)$ where

$$\begin{aligned} S_1^* &= \frac{K(ae_3j_1+d_1e_1j_1-d_1e_3j_1+e_1j_1k_1-\beta_1d_2)}{ae_3j_1} \\ I_1^* &= \frac{-S_1^*e_1j_1+d_2}{j_1e_3} \\ S_2^* &= \frac{S_1^*\beta_1-d_1-k_1}{e_3} \end{aligned}$$

This equilibrium point is feasible if $ae_3j_1 + d_1e_1j_1e_1j_1k_1 > d_1e_3j_1 + \beta_1d_2$ and $\frac{d_1+k_1}{\beta_1} < S_1^* < \frac{d_2}{e_1j_1}$.

(xvi) Coexistence equilibrium point with disease in both species $(S_1,I_1,S_2,I_2) = (S_1^*,I_1^*,S_2^*,I_2^*)$ where:

$$\begin{aligned} S_1^* &= \frac{K\big(-\beta_1\alpha_1{\beta_2}^2+(((a-d_1)(j_2-j_1)e_4-e_2\alpha_1j_1-\beta_1(d_2+k_2))e_3+e_4(e_1\alpha_1j_2+\beta_1d_2))\beta_2-(e_1e_4-e_2e_3)(-j_1(d_2+k_2)e_3+j_2e_4d_2)\big)}{-K{\beta_1}^2{\beta_2}^2+((a(j_2-j_1)e_4-K\beta_1e_2(j_2+j_1))e_3+K\beta_1e_1e_4(j_2+j_1))\beta_2-Kj_2j_1(e_1e_4-e_2e_3)^2} \\ I_1^* &= \frac{-S_1^*e_1e_4j_1+S_1^*e_2e_3j_2+S_1^*\beta_1\beta_2-\beta_2d_1-\beta_2k_1-d_2e_3+d_2e_4-e_3k_2}{e_4e_3(-j_2+j_1)} \\ S_2^* &= \frac{-I_1^*e_4j_2-S_1^*e_2j_2+d_2+k_2}{\beta_2} \\ I_2^* &= \frac{I_1^*e_3j_1+S_1^*e_1j_1-d_2}{\beta_2} \end{aligned} \tag{4.6}$$

With $\alpha_1 = d_1 + k_1$.

This equilibrium point is feasible when S_1^*, I_1^*, S_2^* and I_2^* are positive.

The Jacobian matrix is given below:

$$\begin{bmatrix} a\left(1-\frac{2S_1}{K}\right)-I_1\beta_1-S_2e_1-I_2e_2-d_1 & -S_1\beta_1 & -S_1e_1 & -S_1e_2 \\ I_1\beta_1 & -I_2e_4+S_1\beta_1-S_2e_3-d_1-k_1 & -I_1e_3 & -I_1e_4 \\ j_1S_2e_1 & j_1S_2e_3 & j_1\left(I_1e_3+S_1e_1\right)-I_2\beta_2-d_2 & -S_2\beta_2 \\ j_2I_2e_2 & j_2I_2e_4 & I_2\beta_2 & j_2\left(I_1e_4+S_1e_2\right)+S_2\beta_2-d_2-k_2 \end{bmatrix}$$

To assess stability, we will focus on four of the equilibrium points, namely the equilibrium points corresponding to mutual extinction, coexistence without disease, predator extinction as well as the coexistence with disease in both species.

1. Trivial Equilibrium point (Mutual extinction)

The trivial equilibrium point represents the extinction of both prey and predator species in eco-epidemiology, an undesirable outcome. Knowing when and under what conditions it is stable allows proactive measures to avoid this extinction, contributing to practical ecological conservation through stability analysis. The Jacobian matrix evaluated at the trivial equilibrium point $(S_1, I_1, S_2, I_2) = (0,0,0,0)$ gives the following:

$$\begin{bmatrix} a-d_1 & 0 & 0 & 0 \\ 0 & -d_1-k_1 & 0 & 0 \\ 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & -d_2-k_2 \end{bmatrix}$$

The eigenvalues of the Jacobian matrix evaluated at the trivial equilibrium point are:

(i) $\lambda_1 = -d_2$

(ii) $\lambda_2 = -d_2 - k_2$

(iii) $\lambda_3 = -d_1 - k_1$

(iv) $\lambda_4 = a - d_1$

Since all the parameters are positive then λ_1, λ_2 and λ_3 are all negative. λ_4 is negative when $a < d_1$.

Therefore, when $a < d_1$ all the eigenvalues will be real and negative which means the trivial equilibrium point is stable. When $a > d_1$ then the equilibrium point is unstable.

2. Disease-free Equilibrium point

The equilibrium point is $(S_1, I_1, S_2, I_2) = (S_1^*, 0, S_2^*, 0)$ where $S_1^* = \frac{d_2}{e_1 j_1}$ and $S_2^* = \frac{a}{e_1}\left(1 - \frac{d_2}{j_1 e_1 K} - \frac{d_1}{a}\right)$. The Jacobian matrix evaluated at this equilibrium point is given below:

$$\begin{bmatrix} -\frac{ad_2}{e_1 j_1 K} & -\frac{\beta_1 d_2}{e_1 j_1} & -\frac{d_2}{j_1} & -\frac{d_2 e_2}{e_1 j_1} \\ 0 & \frac{-K j_1 (d_1+k_1) {e_1}^2 - K(e_3(a-d_1) j_1 - \beta_1 d_2) e_1 + a d_2 e_3}{{e_1}^2 j_1 K} & 0 & 0 \\ \frac{K j_1 (a-d_1) e_1 - a d_2}{e_1 K} & \frac{e_3 (K j_1 (a-d_1) e_1 - a d_2)}{K {e_1}^2} & 0 & -\frac{(K j_1 (a-d_1) e_1 - a d_2) \beta_2}{{e_1}^2 j_1 K} \\ 0 & 0 & 0 & \frac{-K j_1 (d_2+k_2) {e_1}^2 + (j_1 (a-d_1) \beta_2 + d_2 e_2 j_2) K e_1 - a \beta_2 d_2}{{e_1}^2 j_1 K} \end{bmatrix}$$

Two obvious eigenvalues can be extracted from the Jacobian matrix namely:

(i) $\lambda_1 = \frac{K a \beta_2 e_1 j_1 - K \beta_2 d_1 e_1 j_1 - d_2 {e_1}^2 j_1 K + d_2 e_2 j_2 e_1 K - k_2 {e_1}^2 j_1 K - a \beta_2 d_2}{{e_1}^2 j_1 K}$

(ii) $\lambda_2 = -\frac{K a e_1 e_3 j_1 + d_1 {e_1}^2 j_1 K - K d_1 e_1 e_3 j_1 + k_1 {e_1}^2 j_1 K - \beta_1 d_2 e_1 K - a d_2 e_3}{{e_1}^2 j_1 K}$

In assessing the stability of the equilibrium point,
$\lambda_1 < 0$ when $a\beta_2 j_1 + d_2 e_2 j_2 < \beta_2 d_1 j_1 + d_2 e_1 j_1 + e_1 j_1 k_2 + \frac{a\beta_2 d_2}{K e_1}$
$\lambda_2 < 0$ when $a e_3 + e_1 (j_1 + k_1) > e_3 \left(d_1 + \frac{a d_2}{K e_1 j_1} \right) + \frac{\beta_1 d_2}{j_1}$

And the remaining two eigenvalues can be obtained from the characteristic polynomial

$$\lambda^2 + \frac{a d_2}{K e_1 j_1} \lambda + a d_2 - d_1 d_2 - \frac{a d_2^2}{K e_1 j_1} = 0 \tag{4.7}$$

The characteristic equation (4.7) exhibits either one sign change when $1 < \frac{d_1}{a} + \frac{d_2}{K j_1 e_1}$ or no sign change when $1 > \frac{d_1}{a} + \frac{d_2}{K j_1 e_1}$. According to Descartes' rule of signs, the first condition implies the existence of one positive eigenvalue, while the latter condition indicates the absence of any positive eigenvalues.

In light of this, the disease-free equilibrium point is stable when both eigenvalues, λ_1 and λ_2, are negative, and the condition $1 > \frac{d_1}{a} + \frac{d_2}{K j_1 e_1}$ is met. The latter condition represents feasibility, meaning that if the equilibrium point is feasible, the only additional requirement for stability is that λ_1 and λ_2 are both negative.

3. The Predator-free Equilibrium point

The equilibrium point is $(S_1^*, I_1^*, S_2^*, I_2^*) = \left(\frac{d_1+k_1}{\beta_1}, \frac{Ka\beta_1 - K\beta_1 d_1 - ad_1 - ak_1}{K{\beta_1}^2}, 0, 0\right)$. The Jacobian matrix evaluated at this equilibrium point is given below:

$$\begin{bmatrix} -\frac{a(d_1+k_1)}{K\beta_1} & -d_1-k_1 & -\frac{e_1(d_1+k_1)}{\beta_1} & -\frac{e_2(d_1+k_1)}{\beta_1} \\ \frac{K(-d_1+a)\beta_1-a(d_1+k_1)}{K\beta_1} & 0 & -\frac{e_3(K(-d_1+a)\beta_1-a(d_1+k_1))}{K{\beta_1}^2} & -\frac{e_4(K(-d_1+a)\beta_1-a(d_1+k_1))}{K{\beta_1}^2} \\ 0 & 0 & \frac{-K{\beta_1}^2 d_2+K((-d_1+a)e_3+e_1(d_1+k_1))j_1\beta_1-ae_3j_1(d_1+k_1)}{K{\beta_1}^2} & 0 \\ 0 & 0 & 0 & \frac{-K(d_2+k_2){\beta_1}^2+K((-d_1+a)e_4+e_2(d_1+k_1))j_2\beta_1-ae_4j_2(d_1+k_1)}{K{\beta_1}^2} \end{bmatrix}$$

Two obvious eigenvalues can be extracted from the Jacobian matrix namely:

(i) $\lambda_1 = \frac{-K(d_2+k_2){\beta_1}^2+K((-d_1+a)e_4+e_2(d_1+k_1))j_2\beta_1-ae_4j_2(d_1+k_1)}{K{\beta_1}^2}$

(ii) $\lambda_2 = \frac{-K{\beta_1}^2 d_2+K((-d_1+a)e_3+e_1(d_1+k_1))j_1\beta_1-ae_3j_1(d_1+k_1)}{K{\beta_1}^2}$

In assessing the stability of the equilibrium point,

$\lambda_1 < 0$ when $Kae_4 + Ke_2(d_1+k_1)j_2\beta_1 < K(d_2+k_2){\beta_1}^2 + ae_4j_2(d_1+k_1) + Kd_1e_4$

$\lambda_2 < 0$ when $K\beta_1 j_1(e_1(d_1+k_1)+ae_3) < K\beta_1 d_1e_3j_1 + K{\beta_1}^2 d_2 + ad_1e_3j_1 + ae_3j_1k_1$

The rest of the eigenvalues can be obtained from the characteristic polynomial:

$$\lambda^2 + \frac{(ad_1+ak_1)}{K\beta_1}\lambda + \frac{(K(a-d_1)\beta_1 - a(d_1+k_1))(d_1+k_1)}{K\beta_1} = 0 \tag{4.8}$$

The characteristic equation (4.8) exhibits either one sign change when $1 < \frac{d_1}{a} + \frac{d_1+k_1}{K\beta_1}$ or no sign change when $1 > \frac{d_1}{a} + \frac{d_1+k_1}{K\beta_1}$. According to Descartes' rule of signs, the first condition implies the existence of one positive eigenvalue, while the latter condition indicates the absence of any positive eigenvalues.

The equilibrium point where there are no predators is considered stable when both eigenvalues, λ_1 and λ_2, are negative and the condition $\frac{d_1+k_1}{K\beta_1} + \frac{d_1}{a} < 1$ is satisfied. This latter condition indicates feasibility, implying that if the equilibrium point is feasible, the only additional requirement for stability is that both λ_1 and λ_2 are negative.

4. <u>The coexistence equilibrium point with disease in both species</u>

   The equilibrium point is $(S_1^*, I_1^*, S_2^*, I_2^*)$ with S_1^*, I_1^*, S_2^* and I_2^* as stated in (4.6).

   The Jacobian matrix evaluated at this equilibrium point is given below:

$$\begin{bmatrix} J_{11} & -\beta_1 S_1^* & -e_1 S_1^* & -e_2 S_1^* \\ \beta_1 I_1^* & J_{22} & -e_4 I_1^* & -e_3 I_1^* \\ j_1 S_2^* e_1 & j_1 S_2^* e_3 & J_{33} & -\beta_2 S_2^* \\ j_2 I_2^* e_2 & j_2 I_2^* e_4 & \beta_2 I_2^* & J_{44} \end{bmatrix}$$

   The variables J_{ij} used in the Jacobian matrix description is given below:

Table 4.2: The variables J_{ij} used in the Jacobian matrix

Variable	**Description**
J_{11}	$a - 2\frac{aS_1^*}{K} - \beta_1 I_1^* - S_2^* e_1 - e_2 I_2^* - d_1$
J_{22}	$-e_3 I_2^* + \beta_1 S_1^* - e_4 S_2^* - d_1 - k_1$
J_{33}	$j_1 \left(e_3 I_1^* + e_1 S_1^*\right) - \beta_2 I_2^* - d_2$
J_{44}	$j_2 \left(e_4 I_1^* + e_2 S_1^*\right) + \beta_2 S_2^* - d_2 - k_2$

The stability of the equilibrium point is assessed using the Routh-Hurwitz criterion.

The characteristic polynomial is given by $\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$

where:

$$\begin{aligned} A = & \quad -J_{44} - J_{33} - J_{22} - J_{11} \\ B = & \quad I_1 S_1^* {\beta_1}^2 + I_2^* S_2^* {\beta_2}^2 + I_1^* J_{32} e_3 + I_1^* J_{42} e_4 + J_{31} S_1^* e_1 + J_{41} S_1^* e_2 + J_{11} J_{22} + J_{11} J_{33} + J_{11} J_{44} + \\ & \quad J_{22} J_{33} + J_{22} J_{44} + J_{33} J_{44} \end{aligned}$$

$$\begin{aligned}
C = \; & I_1^*I_2^*J_{32}\beta_2e_4 - I_1^*J_{31}S_1^*\beta_1e_3 + I_1^*J_{32}S_1^*\beta_1e_1 - I_1^*J_{33}S_1^*{\beta_1}^2 - I_1^*J_{41}S_1^*\beta_1e_4 + I_1^*J_{42}S_1^*\beta_1e_2 - \\
& I_1^*J_{42}S_2^*\beta_2e_3 - I_1^*J_{44}S_1^*{\beta_1}^2 - I_2^*J_{11}S_2^*{\beta_2}^2 - I_2^*J_{22}S_2^*{\beta_2}^2 + I_2^*J_{31}S_1^*\beta_2e_2 - J_{41}S_1^*S_2^*\beta_2e_1 - \\
& I_1^*J_{11}J_{32}e_3 - I_1^*J_{11}J_{42}e_4 - I_1^*J_{32}J_{44}e_3 - I_1^*J_{33}J_{42}e_4 - J_{22}J_{31}S_1^*e_1 - J_{22}J_{41}S_1^*e_2 - \\
& J_{31}J_{44}S_1^*e_1 - J_{33}J_{41}S_1^*e_2 - J_{11}J_{22}J_{33} - J_{11}J_{22}J_{44} - J_{11}J_{33}J_{44} - J_{22}J_{33}J_{44} \\
D = \; & I_2^*J_{11}J_{22}S_2^*{\beta_2}^2 + I_1^*J_{11}J_{33}J_{42}e_4 + J_{22}J_{33}J_{41}S_1^*e_2 + I_1^*J_{33}J_{44}S_1^*{\beta_1}^2 + I_1^*J_{11}J_{32}J_{44}e_3 + \\
& J_{22}J_{31}J_{44}S_1^*e_1 + J_{11}J_{22}J_{33}J_{44} - I_1^*I_2^*J_{31}S_1^*\beta_1\beta_2e_4 + I_1^*I_2^*J_{32}S_1^*\beta_1\beta_2e_2 + I_1^*J_{41}S_1^*S_2^*\beta_1\beta_2e_3 - \\
& I_1^*J_{42}S_1^*S_2^*\beta_1\beta_2e_1 - I_1^*J_{31}J_{42}S_1^*e_2e_3 - I_1^*J_{32}J_{41}S_1^*e_1e_4 + I_1^*I_2^*S_1^*S_2^*{\beta_1}^2{\beta_2}^2 + I_1^*J_{31}J_{42}S_1^*e_1e_4 + \\
& I_1^*J_{32}J_{41}S_1^*e_2e_3 - I_1^*I_2^*J_{11}J_{32}\beta_2e_4 + I_1^*J_{11}J_{42}S_2^*\beta_2e_3 + I_1^*J_{31}J_{44}S_1^*\beta_1e_3 - I_1^*J_{32}J_{44}S_1^*\beta_1e_1 + \\
& I_1^*J_{33}J_{41}S_1^*\beta_1e_4 - I_1^*J_{33}J_{42}S_1^*\beta_1e_2 - I_2^*J_{22}J_{31}S_1^*\beta_2e_2 + J_{22}J_{41}S_1^*S_2^*\beta_2e_1
\end{aligned}$$

The Routh Array is given below:

$$\begin{bmatrix}
1 & B & D & \lambda^4 \\
A & C & 0 & \lambda^3 \\
\frac{AB-C}{A} & D & 0 & \lambda^2 \\
-\frac{A^2D-ABC+C^2}{BA-C} & 0 & 0 & \lambda \\
D & 0 & 0 & 1
\end{bmatrix}$$

The coexistence equilibrium point will be stable if:

(i) $A > 0$;

(ii) $AB - C > 0$;

(iii) $ABC - A^2D - C^2 > 0$;

(iv) $D > 0$.

The complexity of the expressions in the Routh-Hurwitz array described in this analysis makes it difficult to confirm explicitly.

4.2 Model with human intervention in the prey only

4.2.1 Formulation of the model

In this model, treatment and vaccination are given in the prey species. The model is based on the following assumptions:

1. The prey populations grows logistically. In the absence of preys, the population of predators will die out. The presence of the predator species has a negative impact on the growth of the prey.

2. Only the healthy population can reproduce.

3. There is disease in both species and the infection can be transmitted from prey to predator through the process of predation.

4. In both species, the susceptible population becomes infected by coming into contact with the infected population.

5. The prey population is divided into 4 classes: susceptible prey (S_1), infected prey (I_1), vaccinated prey (V_1) and prey under treatment (T_1). On the other hand, the predator population is divided into 2 classes: susceptible predators (S_2) and infected predators (I_2).

6. The infected population does not recover without treatment and there is a natural death rate in all classes as well as a disease induced death rate in both infected populations.

7. After successful treatment, the prey can still be infected at a later stage.

8. Only the susceptible prey are vaccinated which provides partial immunity, and only the infected prey receives treatment.

9. The infected prey is more likely to be caught compared to the healthy ones, and likewise infected predators are less efficient at catching prey than the healthy predators.

The following model can then be derived:

$$\begin{aligned}
S_1' &= \underbrace{aS_1\left(1-\frac{S_1}{K}\right)}_{\text{growth}}\underbrace{-\beta_1 S_1 I_1}_{\text{infection}}\underbrace{-e_1 S_1 S_2 - e_2 S_1 I_2}_{\text{predation}} + \underbrace{\varepsilon T_1}_{\text{treated}}\underbrace{+gV_1 - fS_1}_{\text{vaccination}}\underbrace{-d_1 S_1}_{\text{mortality}}\\
I_1' &= \underbrace{\beta_1 S_1 I_1}_{\text{infection}}\underbrace{-e_3 I_1 S_2 - e_4 I_1 I_2}_{\text{predation}}\underbrace{-hI_1}_{\text{treatment}}\underbrace{-(d_1+k_1)I_1}_{\text{mortality}}\\
V_1' &= \underbrace{fS_1 - gV_1}_{\text{vaccination}}\underbrace{-e_5 V_1 S_2 - e_6 V_1 I_2}_{\text{predation}}\underbrace{-d_1 V_1}_{\text{mortality}}\\
T_1' &= \underbrace{hI_1 - \varepsilon T_1}_{\text{treatment}}\underbrace{-e_7 T_1 S_2 - e_8 T_1 I_2}_{\text{predation}}\underbrace{-(d_1+\iota k_1)T_1}_{\text{mortality}}\\
S_2' &= \underbrace{(e_1 S_1 + e_3 I_1 + e_5 V_1 + e_7 T_1)j_1 S_2}_{\text{predation}}\underbrace{-\beta_2 S_2 I_2}_{\text{infection}}\underbrace{-d_2 S_2}_{\text{mortality}}\\
I_2' &= \underbrace{(e_2 S_1 + e_4 I_1 + e_6 V_1 + e_8 T_1)j_2 I_2}_{\text{predation}}\underbrace{+\beta_2 S_2 I_2}_{\text{infection}}\underbrace{-(d_2+k_2)I_2}_{\text{mortality}}
\end{aligned}\tag{4.9}$$

The parameters' description is given below:

Table 4.4: Model 2 description of parameters

Parameter	**Description**
a	Intrinsic growth rate for prey
$\beta_{1,2}$	Infection rate in the prey and predator respectively
e_i	Predation rate with, $e_{susceptible} < e_{infected}$ for the same predator.
$d_{1,2}$	Natural death rate of the prey and predator respectively
ε	Recovery rate of treated prey
f	Vaccination rate
g	Rate at which vaccine wanes
h	Treatment rate
ι	Factor of disease induced death rate. $\iota \in [0,1]$, $\iota = 0$ indicates that the treatment is 100%

	effective, while $\iota = 1$ signifies that the treatment has no effect.
$j_{1,2}$	Conversion rate of predation for susceptible and infected predators respectively
K	Carrying capacity of the prey
$k_{1,2}$	Disease induced death rate for prey and predators respectively

The flow chart of the above model is given below in figure 4.2:

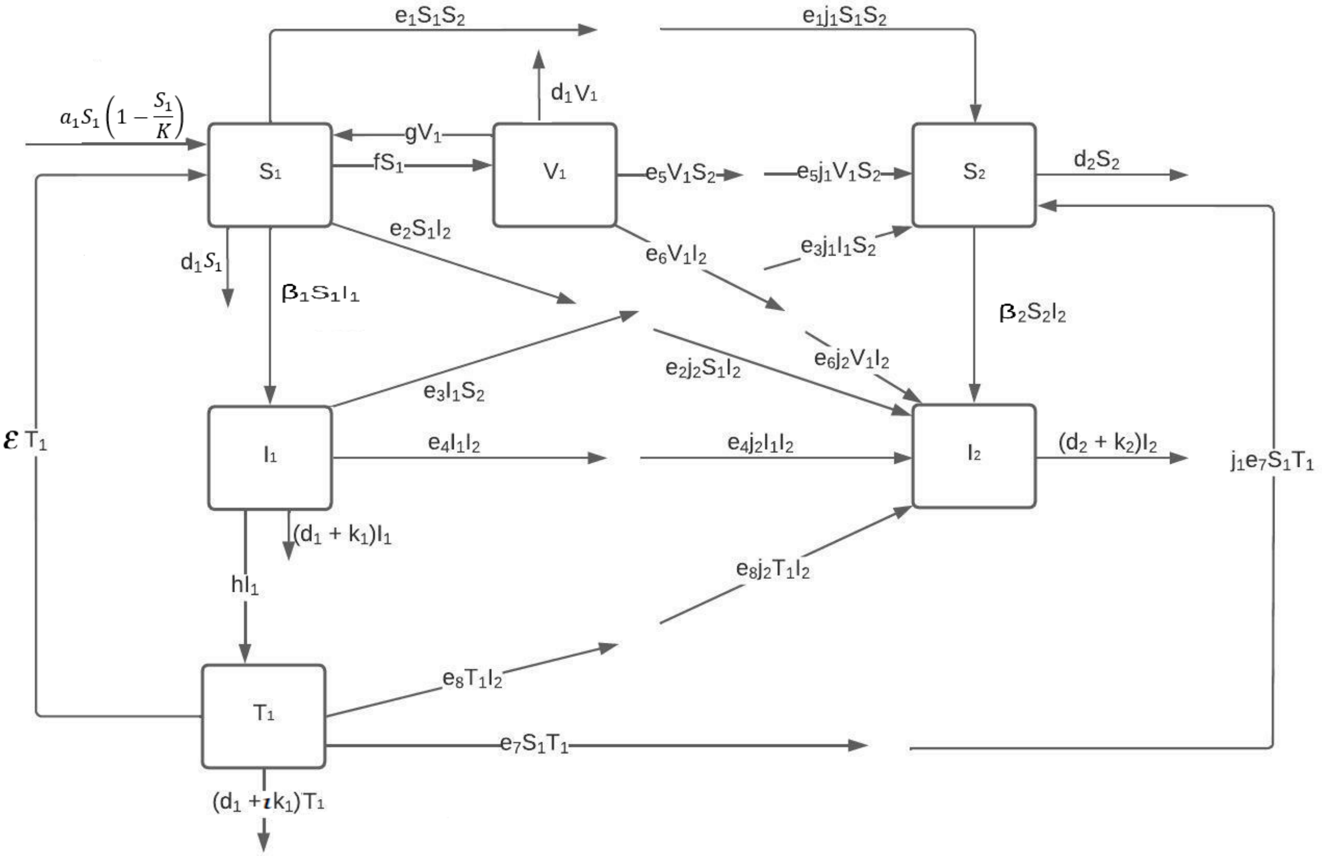

Figure 4.2: Flowchart of model 2 with human intervention in the prey only

4.2.2 Qualitative analysis of the model

4.2.2.1 Positivity of solutions

In this section we show that under the given conditions all the solutions are positive, following a similar proof to that of [4].

Theorem 8 (Positivity)**.** *Let* $S_1(0) \geq 0$, $I_1(0) \geq 0$, $V_1(0) \geq 0$, $T_1(0) \geq 0$, $S_2(0) \geq 0$, $I_2(0) \geq 0$. *The solutions of (4.9) are non-negative for all* $t \geq 0$.

Proof. This will be shown in cases by contradiction.

Case 1. Let t_1 be such that $S_1(t_1) = 0$ and suppose that $S_1^{'}(t_1) < 0$ with $I_1(t) > 0$, $V_1(t) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$ for $0 < t < t_1$. In this case, substituting these values into (4.9), we get $S_1^{'}(t_1) = eT_1(t_1) > 0$, which is a contradiction to the assumption that $S_1^{'}(t_1) < 0$.

Case 2. Let t_2 be such that $I_1(t_2) = 0$ and suppose that $I_1^{'}(t_2) < 0$ with $S_1(t) > 0$, $V_1(t) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$ for $0 < t < t_2$. In this case substituting these values into (4.9) we get $I_1^{'}(t_2) = 0$, which is a contradiction to the assumption that $I_1^{'}(t_2) < 0$.

Case 3. Let t_3 be such that $V_1(t_3) = 0$ and suppose that $V_1^{'}(t_3) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$ for $0 < t < t_3$. In this case, substituting these values into (4.9), we get $V_1^{'}(t_3) = fS_1(t_3) > 0$, which is a contradiction to the assumption that $V_1^{'}(t_3) < 0$.

Case 4. Let t_4 be such that $T_1(t_4) = 0$ and suppose that $T_1^{'}(t_4) < 0$ with $S_1(t) > 0$, $V_1(t) > 0$, $I_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$ for $0 < t < t_4$. In this case, substituting these values into (4.9), we get $T_1^{'}(t_4) = hI_1(t_4) > 0$, which is a contradiction to the assumption that $T_1^{'}(t_4) < 0$.

Case 5. Let t_5 be such that $S_2(t_5) = 0$ and suppose that $S_2^{'}(t_5) < 0$ with $S_1(t) > 0$, $V_1(t) > 0$,

$I_1(t) > 0$, $T_1(t) > 0$, $I_2(t) > 0$ for $0 < t < t_5$. In this case, substituting these values into (4.9), we get $S_2^{'}(t_5) = 0$, which is a contradiction to the assumption that $S_2^{'}(t_5) < 0$.

Case 6. Let t_6 be such that $I_2(t_6) = 0$ and suppose that $I_2^{'}(t_6) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $V_1(t) > 0$, $T_1(t) > 0$, $S_2(t) > 0$ for $0 < t < t_6$. In this case, substituting these values into (4.9), we get $I_2^{'}(t_6) = 0$, which is a contradiction to the assumption that $I_2^{'}(t_6) < 0$.

Hence, all the solutions of (4.9) are non-negative. □

4.2.2.2 Boundedness of solutions

In this section, we demonstrate that under the given conditions, all solutions are bounded above. To establish the boundedness of each population size, it suffices to show that the total population size is bounded. We follow a proof similar to that presented in [12].

Theorem 9 (Boundedness)**.** *Assume that $j_1, j_2 < 1$, then all solutions of the system (4.9) which initiate in $\mathbb{R}_+^6 = \{(S_1, I_1, V_1, T_1, S_2, I_2),\ S_1 \geq 0,\ I_1 \geq 0,\ V_1 \geq 0,\ T_1 \geq 0,\ S_2 \geq 0,\ I_2 \geq 0\}$ are bounded.*

Proof. Let $N = S_1 + I_1 + V_1 + T_1 + S_2 + I_2$ then $\frac{dN}{dt} = S_1' + I_1' + V_1' + T_1' + S_2' + I_2'$.

$$\begin{aligned}\frac{dN}{dt} = \;& aS_1\left(1 - \tfrac{S_1}{k}\right) + (j_1 - 1)\,e_1 S_1 S_2 + (j_2 - 1)e_2 S_1 I_2 + (j_1 - 1)\,e_3 I_1 S_2 + (j_2 - 1)e_4 I_1 I_2 - d_1 S_1 - (d_1 + k_1)I_1 \\ & - d_2 S_2 - (d_2 + k_2)I_2 + (j_1 - 1)e_5 V_1 S_2 + (j_2 - 1)e_6 V_1 T_2 - d_1 V_1 + (j_1 - 1)e_7 T_1 S_2 + (j_2 - 1)e_8 T_1 T_2 \\ & - (d_1 + \iota k_1)T_1\end{aligned}$$

since $j_1, j_2 < 1$ we have $(j_1 - 1) < 0$, and $(j_2 - 1) < 0$. Therefore:

$$\begin{aligned}\frac{dN}{dt} \leq \;& aS_1\left(1 - \tfrac{S_1}{k}\right) - d_1 S_1 - (d_1 + k_1)I_1 - d_1 V_1 - (d_1 + \iota k_1)T_1 - d_2 S_2 - (d_2 + k_2)I_2 \\ \leq \;& aS_1 - \tfrac{aS_1^2}{k} - d_1 S_1 - (d_1 + k_1)I_1 - d_1 V_1 - (d_1 + \iota k_1)T_1 - d_2 S_2 - (d_2 + k_2)I_2 \\ \leq \;& aS_1 - (d_1 S_1 + (d_1 + k_1)I_1 + d_1 V_1 + (d_1 + \iota k_1)T_1 + d_2 S_2 + (d_2 + k_2)I_2) \\ \leq \;& \mu - \eta N\end{aligned}$$

where: $\mu = a\alpha$, $\alpha = \max\{S_1(0), k\}$ and $\eta = \min\{d_1, d_2\}$.

We then have: $\frac{dN}{dt} + \eta N \leq \alpha$ which is a linear first order differential equation. The integration factor is given by $e^{\int \eta dt} = e^{\eta t}$.

$e^{\eta t}\frac{dN}{dt} + \eta e^{\eta t} N \leq \alpha e^{\eta t}$

$\frac{d}{dt}(Ne^{\eta t}) \leq \alpha e^{\eta t}$

$Ne^{\eta t} \leq \frac{\alpha}{\eta} e^{\eta t} + c$

Given the initial condition $N(0) = N_0 \geq 0$, we have $N \leq \frac{\alpha}{\eta} + \left(N_0 + \frac{\alpha}{\eta}\right) e^{-\eta t}$.

As $t \to \infty$, the exponential term $e^{-\eta t} \to 0$. Consequently, we can deduce that $N \leq \frac{\alpha}{\eta}$. Hence all the solutions of (4.9) are bounded and are confined to the region $\Omega = \left\{(S_1, I_1, V_1, T_1, S_2, I_2) \in \mathbb{R}^6_+ : N \leq \frac{\alpha}{\eta}\right\}$.

$\square$

4.2.2.3 Stability of the equilibrium points

The equilibrium points of our system are obtained by setting the right-hand side of (4.9) to zero. When all the components $(S_1, I_1, V_1, T_1, S_2, I_2)$ are zero, the equations remain valid, and we obtain the trivial equilibrium point (0,0,0,0,0,0). Moreover, for the disease-free equilibrium, we only require I_1^* and I_2^* to be zero. It then follows that

$$\begin{aligned}
&aS_1^*\left(1 - \frac{S_1^*}{K}\right) - e_1 S_1^* S_2^* + \varepsilon T_1^* + gV_1^* - fS_1^* - d_1 S_1^* = 0 \\
&fS_1^* - gV_1^* - e_5 V_1^* S_2^* - d_1 V_1^* = 0 \\
&-\varepsilon T_1^* - e_7 T_1^* S_2^* - (d_1 + \imath k_1) T_1^* = 0 \\
&(e_1 S_1^* + e_5 V_1^* + e_7 T_1^*) j_1 S_2^* - d_2 S_2^* = 0
\end{aligned} \tag{4.10}$$

From the third equation of (4.10), it follows that either $T_1^* = 0$ or $-\varepsilon - e_7 S_2 - (d_1 + \imath k_1) = 0$. Since all parameters and S_2 are positive, we can conclude that $T_1^* = 0$. As a result, (4.10) reduces to:

$$\begin{aligned}
&aS_1^*\left(1-\frac{S_1^*}{K}\right) - e_1S_1^*S_2^* + gV_1^* - fS_1^* - d_1S_1^* = 0\\
&fS_1^* - S_2^*V_1^*e_5 - gV_1^* - V_1^*d_1 = 0\\
&j_1S_2^*\left(e_1S_1^* + e_5V_1^*\right) - d_2S_2^* = 0
\end{aligned} \tag{4.11}$$

The disease-free equilibrium point is then given by $(S_1, I_1, V_1, T_1, S_2, I_2) = (S_1^*, 0, V_1^*, 0, S_2^*, 0)$ where S_1^* is a positive root, when it exists, of the equation: $S_1^3 + AS_1^2 + BS_1 + C = 0$.

where:

$$\begin{aligned}
A &= \frac{e_1((a-d_1)e_5+d_1e_1)K{j_1}^2+ad_2e_5j_1}{-ae_1e_5{j_1}^2}\\
B &= \frac{K((a-f-d_1)e_5-e_1(g-d_1))d_2j_1}{ae_1e_5{j_1}^2}\\
C &= \frac{Kg{d_2}^2}{ae_1e_5{j_1}^2}
\end{aligned}$$

According to Descartes' rule of signs, because there are zero or two sign changes regardless of whether A and B are positive or negative, there are either 2 or 0 positive roots. By using theorem (5), we have:

$S_1^* = S + T - \frac{A}{3}$,

where:

$$S = \sqrt[3]{-\frac{1}{27}A^2 + \frac{1}{6}AB - \frac{7C}{54} + \frac{\sqrt{-4A^6+36A^4B+4A^4-36A^3B-27A^2B^2+28A^2C-126ABC+108B^3+49C^2}}{54}}$$

$$T = \sqrt[3]{-\frac{1}{27}A^2 + \frac{1}{6}AB - \frac{7C}{54} - \frac{\sqrt{-4A^6+36A^4B+4A^4-36A^3B-27A^2B^2+28A^2C-126ABC+108B^3+49C^2}}{54}}$$

Descartes' rule of signs states in this case that we have 0 or 2 positive roots. Hence, the given S_1^* is likely the negative one.

$$V_1^* = \frac{d_2-S_1^*e_1j_1}{j_1e_5}$$

$$S_2^* = \frac{fS_1^*j_1e_5+gS_1^*e_1j_1+S_1^*d_1e_1j_1-gd_2-d_1d_2}{e_5(d_2-S_1^*e_1j_1)}.$$

When we have two positive solutions (that might be equal in some cases) for the above cubic polynomial, the feasibility of the corresponding equilibrium points requires that S_1^*,V_1^*, $S_2^* > 0$.

The predator-free equilibrium point is given by $(S_1, I_1, V_1, T_1, S_2, I_2) = (S_1^*, I_1^*, V_1^*, T_1^*, 0, 0)$ where:

$$S_1^* = \frac{h+d_1+k_1}{\beta_1}$$

$$I_1^* = \frac{(h+d_1+k_1)(\imath k_1+\varepsilon+d_1)\left(Kag\beta_1+Ka\beta_1 d_1-Kf\beta_1 d_1-Kg\beta_1 d_1-K\beta_1 {d_1}^2-agh-agd_1-agk_1-ahd_1-a{d_1}^2-ad_1k_1\right)}{K(g+d_1){\beta_1}^2\left(h\imath k_1+\imath d_1k_1+\imath {k_1}^2+\varepsilon d_1+\varepsilon k_1+hd_1+{d_1}^2+d_1k_1\right)}$$

$$V_1^* = \frac{f(h+d_1+k_1)}{\beta_1(g+d_1)}$$

$$T_1^* = \frac{h(h+d_1+k_1)\left(Kag\beta_1+Ka\beta_1 d_1-Kf\beta_1 d_1-Kg\beta_1 d_1-K\beta_1 {d_1}^2-agh-agd_1-agk_1-ahd_1-a{d_1}^2-ad_1k_1\right)}{K(g+d_1){\beta_1}^2\left(h\imath k_1+\imath d_1k_1+\imath {k_1}^2+\varepsilon d_1+\varepsilon k_1+hd_1+{d_1}^2+d_1k_1\right)}$$

This equilibrium point is feasible when
$Kag\beta_1 + Ka\beta_1 d_1 > Kf\beta_1 d_1 + Kg\beta_1 d_1 + K\beta_1 {d_1}^2 + agh + agd_1 + agk_1 + ahd_1 + a{d_1}^2 + ad_1k_1$.
The Jacobian matrix is given as follows:

$$\begin{bmatrix} J_{11} & -\beta_1 S_1 & g & \varepsilon & -e_1S_1 & -e_2S_1 \\ \beta_1 I_1 & J_{22} & 0 & 0 & -e_3I_1 & -e_4I_1 \\ f & 0 & J_{33} & 0 & -e_5V_1 & -e_6V_1 \\ 0 & h & 0 & J_{44} & -e_7T_1 & -e_8T_1 \\ j_1S_2e_1 & j_1S_2e_3 & j_1S_2e_5 & j_1S_2e_7 & J_{55} & -\beta_2 S_2 \\ I_2j_2e_2 & I_2j_2e_4 & I_2j_2e_6 & I_2j_2e_8 & \beta_2 I_2 & J_{66} \end{bmatrix}$$

The variables J_{ij} used in the Jacobian matrix description is given below:

Table 4.7: Model 2 description of elements of the Jacobian matrix of the model with human intervention in the prey only

Variable	**Description of variable**
J_{11}	$a\left(1-\frac{S_1}{K}\right)-\frac{aS_1}{K}-\beta_1 I_1-e_1S_2-e_2I_2-f-d_1$
J_{22}	$-e_4I_2+\beta_1S_1-e_3S_2-h-d_1-k_1$
J_{33}	$-I_2e_6-S_2e_5-g-d_1$
J_{44}	$-\imath k_1-e_8I_2-e_7S_2-\varepsilon-d_1$
J_{55}	$j_1\left(e_3I_1+e_1S_1+e_7T_1+e_5V_1\right)-\beta_2I_2-d_2$
J_{66}	$j_2\left(e_4I_1+e_2S_1+e_8T_1+e_6V_1\right)+\beta_2S_2-d_2-k_2$

We then analyse three equilibrium points namely, trivial, disease-free and the predator-free to assess their stability:

1. Trivial Equilibrium point $S_1^*=I_1^*=V_1^*=T_1^*=S_2^*=I_2^*=0$

   The Jacobian matrix evaluated at the trivial equilibrium point $(S_1^*,I_1^*,V_1^*,T_1^*,S_2^*,I_2^*)=(0,0,0,0,0,0)$ gives the following:

$$\begin{bmatrix} a-f-d_1 & 0 & g & \varepsilon & 0 & 0 \\ 0 & -h-d_1-k_1 & 0 & 0 & 0 & 0 \\ f & 0 & -g-d_1 & 0 & 0 & 0 \\ 0 & h & 0 & -\imath k_1-\varepsilon-d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d_2-k_2 \end{bmatrix}$$

   Four eigenvalues can be extracted from the Jacobian matrix namely:

   (i) $\lambda_1=-d_2$

   (ii) $\lambda_2=-d_2-k_2$

   (iii) $\lambda_3=-h-d_1-k_1$

(iv) $\lambda_4 = -\imath k_1 - \varepsilon - d_1$

Since all the parameters are positive then $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$.

The reduced characteristic polynomial is given by:

$$\lambda^2 + (-a + f + g + 2d_1)\lambda - ga - d_1 a + d_1 f + g d_1 + {d_1}^2 = 0 \tag{4.12}$$

The characteristic equation (4.12) exhibits no sign changes when $a < f + g + 2d_1$, and $a(g + d_1) < d_1(f + g + d_1)$. According to Descartes' rule of signs, this implies that there are no positive eigenvalues.

Stability of the trivial equilibrium point requires that all of the following conditions hold: The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$, $a < f + g + 2d_1$, and $a(g + d_1) < d_1(f + g + d_1)$.

2. <u>Disease-free equilibrium point</u>

   The Jacobian matrix evaluated at the disease-free equilibrium point $(S_1, I_1, V_1, T_1, S_2, I_2) = (S_1^*, 0, V_1^*, 0, S_2^*, 0)$ gives the following:

$$\begin{bmatrix} K_{11} & -\beta_1 S_1^* & g & \varepsilon & -e_1 S_1^* & -e_2 S_1^* \\ 0 & K_{22} & 0 & 0 & 0 & 0 \\ f & 0 & K_{33} & 0 & -e_5 V_1^* & -e_6 V_1^* \\ 0 & h & 0 & K_{44} & 0 & 0 \\ j_1 S_2^* e_1 & j_1 S_2^* e_3 & j_1 S_2^* e_5 & j_1 S_2^* e_7 & K_{55} & -\beta_2 S_2^* \\ 0 & 0 & 0 & 0 & 0 & K_{66} \end{bmatrix}$$

   The variables K_{ij} used in the Jacobian matrix evaluated at the disease-free equilibrium point's description is given below:

Table 4.8: Model 2 description of elements, Jacobian matrix in the model with human intervention in the prey only

Variable	**Description of variables**
K_{11}	$a\left(1-\frac{S_1^*}{K}\right)-\frac{aS_1^*}{K}-e_1S_2^*-f-d_1$
K_{22}	$\beta_1S_1^*-e_3S_2^*-h-d_1-k_1$
K_{33}	$-S_2^*e_5-g-d_1$
K_{44}	$-\iota k_1-S_2^*e_7-\varepsilon-d_1$
K_{55}	$j_1\left(e_1S_1^*+e_5V_1^*\right)-d_2$
K_{66}	$j_2\left(e_2S_1^*+e_6V_1^*\right)+\beta_2S_2^*-d_2-k_2$

Three eigenvalues can be extracted from the the Jacobian matrix namely:

(i) $\lambda_1=\beta_1S_1^*-e_3S_2^*-h-d_1-k_1$

(ii) $\lambda_2=-\iota k_1-S_2^*e_7-\varepsilon-d_1$

(iii) $\lambda_3=j_2\left(e_2S_1^*+e_6V_1^*\right)+\beta_2S_2^*-d_2-k_2$

In assessing the stability of the equilibrium point,

$\lambda_1<0$ when $\beta_1S_1^*<e_3S_2^*+h+d_1+k_1$

$\lambda_2<0$ since S_2^* and all the parameters as positive.

$\lambda_3<0$ when $j_2\left(e_2S_1^*+e_6V_1^*\right)+\beta_2S_2^*<d_2+k_2$

The reduced characteristic polynomial is given by:

$$\lambda^3+A\lambda^2+B\lambda+C=0 \tag{4.13}$$

where:

$$A=\quad -\frac{1}{K}\left(\left(j_1\left(S_1^*e_1+V_1^*e_5\right)-e_1S_2^*-S_2^*e_5-d_2+a-f-g-2d_1\right)K-2aS_1^*\right)$$

$$
\begin{aligned}
B = \quad & -\tfrac{1}{K}(KS_1^*S_2^*e_1e_5j_1 + KS_2^*V_1^*e_1e_5j_1 - KaS_1^*e_1j_1 - KaV_1^*e_5j_1 + KfS_1^*e_1j_1 + KfV_1^*e_5j_1 + \\
& KgS_1^*e_1j_1 + KgV_1^*e_5j_1 + 2KS_1^*d_1e_1j_1 - KS_2^{*2}e_1e_5 + 2KV_1^*d_1e_5j_1 + 2aS_1^{*2}e_1j_1 + \\
& 2aS_1^*V_1^*e_5j_1 + KaS_2^*e_5 - KfS_2^*e_5 - KgS_2^*e_1 - KS_2^*d_1e_1 - KS_2^*d_1e_5 - KS_2^*d_2e_1 - KS_2^*d_2e_5 - \\
& 2aS_1^*S_2e_5 + Kag + Kad_1 + Kad_2 - Kfd_1 - Kfd_2 - Kgd_1 - Kgd_2 - Kd_1{}^2 - 2Kd_1d_2 - \\
& 2agS_1^* - 2aS_1^*d_1 - 2aS_1^*d_2) \\
C = \quad & -\tfrac{1}{K}(-KaS_1^*S_2^*e_1e_5j_1 + KS_1^*S_2^*d_1e_1e_5j_1 + KS_2^*V_1^*d_1e_1e_5j_1 + 2aS_1^{*2}S_2^*e_1e_5j_1 - \\
& KagS_1^*e_1j_1 - KagV_1^*e_5j_1 - KaS_1^*d_1e_1j_1 - KaV_1^*d_1e_5j_1 + KfS_1^*d_1e_1j_1 + KfV_1^*d_1e_5j_1 + \\
& KgS_1^*d_1e_1j_1 + KgV_1^*d_1e_5j_1 + KS_1^*d_1{}^2e_1j_1 - KS_2^{*2}d_2e_1e_5 + KV_1^*d_1{}^2e_5j_1 + 2agS_1^{*2}e_1j_1 + \\
& 2agS_1^*V_1^*e_5j_1 + 2aS_1^{*2}d_1e_1j_1 + 2aS_1^*V_1^*d_1e_5j_1 + KaS_2^*d_2e_5 - KfS_2^*d_2e_5 - KgS_2^*d_2e_1 - \\
& KS_2^*d_1d_2e_1 - KS_2^*d_1d_2e_5 - 2aS_1^*S_2^*d_2e_5 + Kagd_2 + Kad_1d_2 - Kfd_1d_2 - Kgd_1d_2 - \\
& Kd_1{}^2d_2 - 2agS_1^*d_2 - 2aS_1^*d_1d_{2)}
\end{aligned}
$$

The Routh array is given by:

$$
\begin{bmatrix}
1 & B & \lambda^3 \\
A & C & \lambda^2 \\
\frac{AB-C}{A} & 0 & \lambda \\
C & 0 & 1
\end{bmatrix}
$$

Stability at the disease-free equilibrium point necessitates that all four conditions are met: the eigenvalues λ_1, λ_2, and λ_3 must be negative, and in addition, $A > 0$, $C > 0$ and $AB > C$ must hold.

3. Predator-free equilibrium point

With intervention in the prey only, it is possible for the predators to go to extinction. In that case $S_2^* = I_2^* = 0$. The Jacobian matrix evaluated at this equilibrium point is given below:

$$\begin{bmatrix} K_{11} & -h-d_1-k_1 & g & \varepsilon & -\frac{e_1(h+d_1+k_1)}{\beta_1} & -\frac{e_2(h+d_1+k_1)}{\beta_1} \\ K_{21} & 0 & 0 & 0 & K_{25} & K_{26} \\ f & 0 & -g-d_1 & 0 & -\frac{e_5 f(h+d_1+k_1)}{\beta_1(g+d_1)} & -\frac{e_6 f(h+d_1+k_1)}{\beta_1(g+d_1)} \\ 0 & h & 0 & -ik_1-\varepsilon-d_1 & K_{45} & K_{46} \\ 0 & 0 & 0 & 0 & K_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{66} \end{bmatrix}$$

The variables K_{ij} used in the Jacobian matrix's description is given below:

Table 4.10: Model 2 description of elements, Jacobian matrix for model with human intervention in the prey only

Variable	**Description of variables**
K_{11}	$\frac{-ad_1{}^4-a\alpha_9 d_1{}^3+\alpha_{11}d_1{}^2+\alpha_{12}d_1-g\left(\left(\iota k_1{}^3+(2h\iota+\varepsilon)k_1{}^2+h^2\iota k_1+h\varepsilon\,(K\beta_1-h)\right)a+K\beta_1 f k_1(h\iota+\iota k_1+\varepsilon)\right)}{\alpha_1}$
K_{21}	$\frac{(\iota k_1+\varepsilon+d_1)\left((-K\beta_1-a)d_1{}^2+(-ak_1+(K\beta_1-g-h)a-K\beta_1(f+g))d_1+ag(K\beta_1-h-k_1)\right)(h+d_1+k_1)}{\alpha_1}$
K_{25}	$-\frac{(\iota k_1+\varepsilon+d_1)\left((-K\beta_1-a)d_1{}^2+(-ak_1+(K\beta_1-g-h)a-K\beta_1(f+g))d_1+ag(K\beta_1-h-k_1)\right)e_3(h+d_1+k_1)}{\alpha_1}$
K_{26}	$-\frac{(\iota k_1+\varepsilon+d_1)\left((-K\beta_1-a)d_1{}^2+(-ak_1+(K\beta_1-g-h)a-K\beta_1(f+g))d_1+ag(K\beta_1-h-k_1)\right)e_4(h+d_1+k_1)}{\alpha_1}$
K_{45}	$-\frac{\left((-K\beta_1-a)d_1{}^2+(-ak_1+(K\beta_1-g-h)a-K\beta_1(f+g))d_1+ag(K\beta_1-h-k_1)\right)e_7(h+d_1+k_1)h}{\alpha_1}$
K_{46}	$-\frac{\left((-K\beta_1-a)d_1{}^2+(-ak_1+(K\beta_1-g-h)a-K\beta_1(f+g))d_1+ag(K\beta_1-h-k_1)\right)e_8(h+d_1+k_1)h}{\alpha_1}$
K_{55}	$\frac{j_1\alpha_3(\alpha_5(\alpha_8-a\alpha_3\alpha_4)e_3+K((k_1(\alpha_4 e_1+e_5 f)\iota+d_1\alpha_4 e_1+e_7\alpha_4 a+(-\alpha_7 e_7+e_5 f)d_1)h+\alpha_6\alpha_5(\alpha_4 e_1+e_5 f))\beta_1-ahe_7\alpha_3\alpha_4)}{\alpha_2}-d_2$
K_{66}	$\frac{j_2((\alpha_8-a\alpha_3\alpha_4)\alpha_5 e_4+((k_1(\alpha_4 e_2+e_6 f)\iota+d_1\alpha_4 e_2+e_8\alpha_4 a+(\alpha_7 e_8+e_6 f)d_1)h+\alpha_6\alpha_5(\alpha_4 e_2+e_6 f))K\beta_1-ahe_8\alpha_3\alpha_4)\alpha_3}{\alpha_2}$

The newly introduced variables are described in the table below.

Table 4.11: Variables explained

Variable	**Description of variables**
α_1	$K\beta_1 (g+d_1) \left({d_1}^2 + ((\imath+1) k_1 + h + \varepsilon) d_1 + k_1 (h\imath + \imath k_1 + \varepsilon)\right)$
α_2	$K{\beta_1}^2 (g+d_1) ((\imath k_1 + d_1) h + (d_1 + k_1) (\imath k_1 + \varepsilon + d_1)) - d_2 - k_2$
α_3	$h + d_1 + k_1$
α_4	$g + d_1$
α_5	$\imath k_1 + \varepsilon + d_1$
α_6	$d_1 + k_1$
α_7	$f + g + d_1$
α_8	$((g+d_1)a - d_1(f+g+d_1)) K\beta_1$
α_9	$(\imath+2) k_1 + g + 2h + \varepsilon$
α_{10}	$(-\imath-2) g + (-2\imath-2) h - 2\varepsilon$
α_{11}	$\left((-2\imath-1) {k_1}^2 + ((-\imath-2) g + (-2\imath-2) h - 2\varepsilon) k_1 + (-2h-\varepsilon) g - h^2\right) a - K\beta_1 (-\varepsilon h + fg)$
α_{12}	$\left(-\imath {k_1}^3 + ((-2\imath-1) g - 2h\imath - \varepsilon) {k_1}^2 + \left(((-2\imath-2) h - 2\varepsilon) g - h^2\imath\right) k_1 - (gh + \varepsilon (K\beta_1 - h)) h\right) a$ $-K\beta_1 (fg (\imath+1) k_1 + ((f-\varepsilon) h + f\varepsilon) g - fh\varepsilon)$

Two eigenvalues K_{55} and K_{66} can be extracted from the Jacobian matrix:

(i) $\lambda_1 = \frac{j_1\alpha_3(\alpha_5(\alpha_8 - a\alpha_3\alpha_4)e_3 + K((k_1(\alpha_4 e_1 + e_5 f)\imath + d_1\alpha_4 e_1 + e_7\alpha_4 a + (-\alpha_7 e_7 + e_5 f)d_1)h + \alpha_6\alpha_5(\alpha_4 e_1 + e_5 f))\beta_1 - ahe_7(\alpha_3)\alpha_4)}{\alpha_2} - d_2$

(ii) $\lambda_2 = \frac{j_2((\alpha_8 - a\alpha_3\alpha_4)\alpha_5 e_4 + ((k_1(\alpha_4 e_2 + e_6 f)\imath + d_1\alpha_4 e_2 + e_8\alpha_4 a + (\alpha_7 e_8 + e_6 f)d_1)h + \alpha_6\alpha_5(\alpha_4 e_2 + e_6 f))K\beta_1 - ahe_8\alpha_3\alpha_4)\alpha_3}{\alpha_2}$

The reduced characteristic polynomial is given by:

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0 \tag{4.14}$$

where:

$A = \quad \imath k_1 + \varepsilon + g - K_{11} + 2d_1$

$$
\begin{aligned}
B = \quad & g\imath k_1 - \imath K_{11}k_1 + \imath d_1k_1 + \varepsilon g - \varepsilon K_{11} + \varepsilon d_1 - fg - gK_{11} + gd_1 + hK_{21} + 2K_{11}d_1 + K_{21}d_1 + \\
& K_{21}k_1 + {d_1}^2 \\
C = \quad & -fg\imath k_1 - g\imath K_{11}k_1 + h\imath K_{21}k_1 - \imath K_{11}d_1k_1 + \imath K_{21}d_1k_1 + \imath K_{21}{k_1}^2 - \varepsilon fg - \varepsilon gK_{11} - \\
& \varepsilon K_{11}d_1 + \varepsilon K_{21}d_1 + \varepsilon K_{21}k_1 - fgd_1 + ghK_{21} - gK_{11}d_1 + gK_{21}d_1 + gK_{21}k_1 + 2hK_{21}d_1 - \\
& K_{11}{d_1}^2 + 2K_{21}{d_1}^2 + 2K_{21}d_1k_1 \\
D = \quad & gh\imath K_{21}k_1 + g\imath K_{21}d_1k_1 + g\imath K_{21}{k_1}^2 + h\imath K_{21}d_1k_1 + \imath K_{21}{d_1}^2k_1 + \imath K_{21}d_1{k_1}^2 + \varepsilon gK_{21}d_1 + \\
& \varepsilon gK_{21}k_1 + \varepsilon K_{21}{d_1}^2 + \varepsilon K_{21}d_1k_1 + ghK_{21}d_1 + gK_{21}{d_1}^2 + gK_{21}d_1k_1 + hK_{21}{d_1}^2 + K_{21}{d_1}^3 + \\
& K_{21}{d_1}^2k_1
\end{aligned}
$$

The Routh array is given by:

$$
\begin{bmatrix}
1 & B & D & \lambda^4 \\
A & C & 0 & \lambda^3 \\
\frac{AB-C}{A} & D & 0 & \lambda^2 \\
\frac{ABC-A^2D-C^2}{AB-C} & 0 & 0 & \lambda \\
D & 0 & 0 & 1
\end{bmatrix}
$$

Stability at the predator-free equilibrium point requires that all five conditions be met: the eigenvalues λ_1 and λ_2 must be negative, and in addition, $AB > C$, $ABC > A^2D + C^2$, and $D > 0$.

4.3 Model with human Intervention in the predator only

4.3.1 Formulation of the model

In this model, treatment and vaccination are provided in the predator species. The model is based on the following assumptions:

1. The prey populations grows logistically. In the absence of preys, the population of predators will die out. The presence of the predator species has a negative impact on the growth of the

prey.

2. Only the healthy population can reproduce.
3. There is disease in both species and infection can be transmitted from prey to predator through the process of predation.
4. In both populations, the susceptible population becomes infected by coming into contact with the infected population.
5. The prey population is divided into 2 classes: susceptible prey (S_1) and infected prey (I_1). Likewise the predator population is divided into 4 classes: susceptible predators (S_2) and infected predators (I_2), vaccinated predators (V_2) and predators under treatment (T_2).
6. The infected predators never recover without treatment.
7. There is a natural death rate in all classes as well as a disease induced death rate in both infected populations.
8. Only the susceptible predators are vaccinated which provides partial immunity, and only the infected predators receives treatment.
9. After successful treatment, the predators can still be contaminated at a later stage (partial immunity).
10. The infected prey is more likely to be caught compared to the healthy ones, and likewise infected predators are less efficient at catching prey than the healthy predators.

The following model can then be derived:

$$
\begin{aligned}
S_1' &= \underbrace{aS_1\left(1-\frac{S_1}{K}\right)}_{\text{growth}}\underbrace{-\beta_1S_1I_1}_{\text{infection}}\underbrace{-e_1S_1S_2-e_2S_1I_2-e_3S_1V_2-e_4S_1T_2}_{\text{predation}}\underbrace{-d_1S_1}_{\text{mortality}}\\
I_1' &= \underbrace{\beta_1S_1I_1}_{\text{infection}}\underbrace{-e_5I_1S_2-e_6I_1I_2-e_7I_1V_2-e_8I_1T_2}_{\text{predation}}\underbrace{-(d_1+k_1)I_1}_{\text{mortality}}\\
S_2' &= \underbrace{(e_1S_1+e_5I_1)j_1S_2}_{\text{predation}}\underbrace{-\beta_2S_2I_2}_{\text{infection}}\underbrace{-fS_2+gV_2}_{\text{vaccination}}\underbrace{+\varepsilon T_2}_{\text{treated}}\underbrace{-d_2S_2}_{\text{mortality}}\\
I_2' &= \underbrace{(e_2S_1+e_6I_1)j_2I_2}_{\text{predation}}\underbrace{+\beta_2S_2I_2}_{\text{infection}}\ \underbrace{-hI_2}_{\text{treatment}}\ \underbrace{-(d_2+k_2)I_2}_{\text{mortality}}\\
V_2' &= \underbrace{(e_3S_1+e_7I_1)j_3V_2}_{\text{predation}}\underbrace{+fS_2-gV_2}_{\text{vaccination}}\underbrace{-d_2V_2}_{\text{mortality}}\\
T_2' &= \underbrace{(e_4S_1+e_8I_1)j_4T_2}_{\text{predation}}+\underbrace{hI_2-\varepsilon T_2}_{\text{treatment}}\underbrace{-(d_2+\iota k_2)T_2}_{\text{mortality}}
\end{aligned}
\tag{4.15}
$$

The parameters' description is given below:

Table 4.13: Model 3 description of parameters

Parameter	**Description**
a	Intrinsic growth rate for prey
$b_{1,2}$	Infection rate in the prey and predator respectively
e_i	Predation rate with, $e_{susceptible} < e_{infected}$ for the same predator.
$d_{1,2}$	Natural death rate of the prey and predator respectively
ε	Recovery rate of treated prey
f	Vaccination rate
g	Rate at which the vaccine wears off
h	Treatment rate
ι	Factor of disease induced death rate

$j_{1,2}$	Conversion rate of predation for susceptible and infected predators respectively
K	Carrying capacity of the prey
$k_{1,2}$	Disease induced death rate for prey and predators respectively

The flowchart of the above model is given below in figure 4.3:

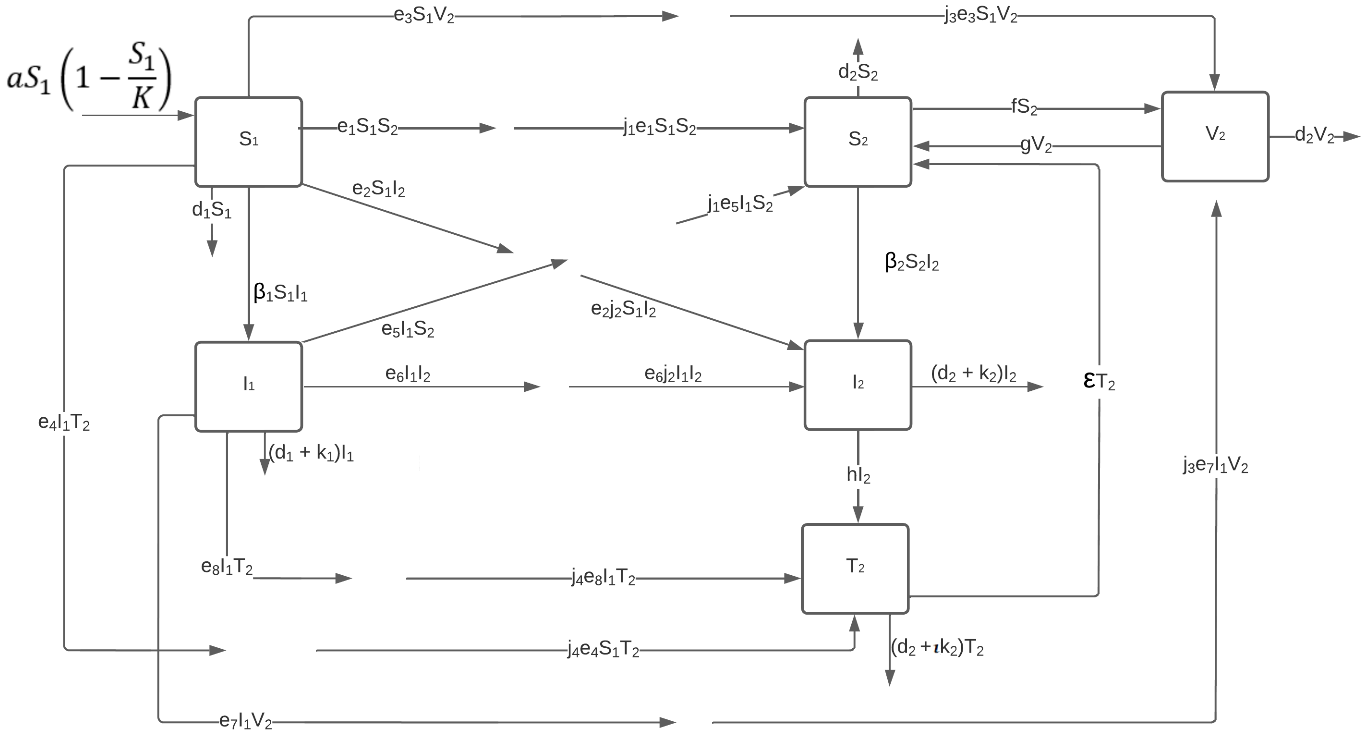

Figure 4.3: Flowchart of Model 3 with human intervention in the predators only

4.3.2 Qualitative analysis of the model

4.3.2.1 Positivity of solutions

In this section we show that under the given conditions all the solutions are positive, following a similar proof to that of [4].

Theorem 10 (Positivity). *Let $S_1(0) \geq 0, I_1(0) \geq 0, S_2(0) \geq 0, I_2(0) \geq 0, V_2(0) \geq 0, T_2(0) \geq 0$. The solutions of (4.15) are non-negative for all $t \geq 0$.*

Proof. This will be shown in cases by contradiction.

Case 1. Let t_1 be such that $S_1(t_1) = 0$ and suppose that $S_1'(t_1) < 0$ with $I_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_1$. In this case, substituting these values into (4.15), we get $S_1'(t_1) = 0$, which is a contradiction to the assumption that $S_1'(t_1) < 0$.

Case 2. Let t_2 be such that $I_1(t_2) = 0$ and suppose that $I_1'(t_2) < 0$ with $S_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_2$. In this case, substituting these values into (4.15), we get $I_1'(t_2) = 0$, which is a contradiction to the assumption that $I_1'(t_2) < 0$.

Case 3. Let t_3 be such that $S_2(t_3) = 0$ and suppose that $S_2'(t_3) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_3$. In this case, substituting these values into (4.15), we get $S_2'(t_3) = gV_1(t_3) > 0$, which is a contradiction to the assumption that $S_2'(t_3) < 0$.

Case 4. Let t_4 be such that $I_2(t_4) = 0$ and suppose that $I_2'(t_4) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $S_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_4$. In this case, substituting these values into (4.15), we get $I_2'(t_4) = 0$, which is a contradiction to the assumption that $I_2'(t_4) < 0$.

Case 5. Let t_5 be such that $V_2(t_5) = 0$ and suppose that $V_2'(t_5) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_5$. In this case, substituting these values into (4.15), we get $V_2'(t_5) = 0$, which is a contradiction to the assumption that $V_2'(t_5) < 0$.

Case 6. Let t_6 be such that $T_2(t_6) = 0$ and suppose that $T_2'(t_6) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$ for $0 < t < t_6$. In this case, substituting these values into (4.15), we get $T_2'(t_6) = hI_2(t_6) > 0$, which is a contradiction to the assumption that $T_2'(t_6) < 0$.

Hence, all the solutions of (4.15) are non-negative. □

4.3.2.2 Boundedness of solutions

In this section, we demonstrate that under the given conditions, all solutions are bounded above. To establish the boundedness of each population size, it suffices to show that the total population size is bounded. We follow a proof similar to that presented in [12].

Theorem 11 (Boundedness)**.** *Assume that $j_1, j_2, j_3, j_4 < 1$, then all solutions of the system (4.15) which initiate in $\mathbb{R}^6_+ = \{(S_1, I_1, S_2, I_2, V_2, T_2),\ S_1 \geq 0,\ I_1 \geq 0,\ S_2 \geq 0,\ I_2 \geq 0,\ V_2 \geq 0,\ T_2 \geq 0\}$ are bounded.*

Proof. Let $N = S_1 + I_1 + S_2 + I_2 + V_2 + T_2$ then $\frac{dN}{dt} = S_1' + I_1' + S_2' + I_2' + V_2' + T_2'$.

$$\begin{aligned}\frac{dN}{dt} = \ & aS_1\left(1 - \tfrac{S_1}{k}\right) + (j_1 - 1)\,e_1S_1S_2 + (j_2 - 1)e_2S_1I_2 + (j_3 - 1)e_3S_1V_2 + (j_4 - 1)e_4S_1T_2 \\ & + (j_1 - 1)\,e_5I_1S_2 + (j_2 - 1)e_6I_1I_2 + (j_3 - 1)e_7I_1V_2 + (j_4 - 1)e_8I_1T_2 - d_1S_1 - (d_1 + k_1)I_1 - d_2S_2 \\ & -(d_2 + k_2)I_2 - d_2V_2 - (d_2 + \imath k_2)T_2\end{aligned}$$

since $j_1, j_2, j_3, j_4 < 1$ we have $(j_1 - 1) < 0$, $(j_2 - 1) < 0$, $(j_3 - 1) < 0$ and $(j_4 - 1) < 0$. Therefore:

$$\begin{aligned}\frac{dN}{dt} \leq \ & aS_1\left(1 - \tfrac{S_1}{k}\right) - d_1S_1 - (d_1 + k_1)I_1 - d_2V_2 - (d_2 + \imath k_2)T_2 - d_2S_2 - (d_2 + k_2)I_2 \\ \leq \ & aS_1 - \tfrac{aS_1^2}{k} - d_1S_1 - (d_1 + k_1)I_1 - d_2V_2 - (d_2 + \imath k_2)T_2 - d_2S_2 - (d_2 + k_2)I_2 \\ \leq \ & aS_1 - (d_1S_1 + (d_1 + k_1)I_1 + d_2V_2 + (d_2 + \imath k_2)T_2 + d_2S_2 + (d_2 + k_2)I_2) \\ \leq \ & \mu - \eta N\end{aligned}$$

where: $\mu = a\alpha$, $\alpha = \max\{S_1(0), k\}$ and $\eta = \min\{d_1, d_2\}$.

We then have: $\frac{dN}{dt} + \eta N \leq \alpha$ which is a linear first order differential equation. The integration factor is given by $e^{\int \eta dt} = e^{\eta t}$.

$e^{\eta t}\frac{dN}{dt} + \eta e^{\eta t}N \leq \alpha e^{\eta t}$

$\frac{d}{dt}\left(Ne^{\eta t}\right) \leq \alpha e^{\eta t}$

$Ne^{\eta t} \leq \frac{\alpha}{\eta} e^{\eta t} + c$

Given the initial condition $N(0) = N_0 \geq 0$, we have $N \leq \frac{\alpha}{\eta} + \left(N_0 + \frac{\alpha}{\eta}\right) e^{-\eta t}$.

As $t \to \infty$, the exponential term $e^{-\eta t} \to 0$. Consequently, we can deduce that $N \leq \frac{\alpha}{\eta}$. Hence all the solutions of (4.15) are bounded and are confined to the region $\Omega = \left\{(S_1, I_1, S_2, I_2, V_2, T_2) \in \mathbb{R}^6_+ : N \leq \frac{\alpha}{\eta}\right\}$.

□

4.3.2.3 Stability of the equilibrium points

The equilibrium points of the system are obtained by setting the right-hand side of (4.15) to zero. When all the components $(S_1, I_1, S_2, I_2, V_2, T_2)$ are zero, the equations remain valid, and we obtain the trivial equilibrium point (0,0,0,0,0,0). Moreover, for the disease-free equilibrium, we only require I_1^* and I_2^* to be zero. It therefore follows:

$$\begin{aligned}
&aS_1^*\left(1 - \frac{S_1^*}{K}\right) - e_1 S_1^* S_2^* - e_3 S_1^* V_2^* - e_4 S_1^* T_2^* - d_1 S_1^* = 0\\
&j_1 S_2^* S_1^* e_1 - f S_2^* + g V_2^* + \varepsilon T_2^* - d_2 S_2^* = 0\\
&f S_2^* - g V_2^* + e_3 j_3 V_2^* S_1^* - d_2 V_2^* = 0\\
&e_4 S_1^* j_4 T_2^* - \varepsilon T_2^* - (d_2 + \iota k_2) T_2^* = 0
\end{aligned} \tag{4.16}$$

From the last equation of (4.16), it follows that $T_2^* = 0$ or $S_1^* = \frac{\varepsilon + d_2 + \iota k_2}{e_4 j_4}$.

In the case when $T_2 = 0$, (4.16) reduces to:

$$\begin{aligned}
&aS_1^*\left(1 - \frac{S_1^*}{K}\right) - e_1 S_1^* S_2^* - e_3 S_1^* V_2^* - d_1 S_1^* = 0\\
&j_1 S_2^* S_1^* e_1 - f S_2^* + g V_2^* - d_2 S_2^* = 0\\
&f S_2^* - g V_2^* + V_2^* j_3 S_1^* e_3 - d_2 V_2^* = 0
\end{aligned} \tag{4.17}$$

The disease-free equilibrium point is then given by $(S_1, I_1, S_2, I_2, V_2, T_2) = (S_1^*, 0, S_2^*, 0, V_2^*, 0)$ where S_1^* is a positive root of the equation:

$$e_1e_3j_1j_3{S_1}^2-\left(fe_3j_3+ge_1j_1+d_2e_1j_1+d_2e_3j_3\right)S_1+fd_2+gd_2+{d_2}^2=0$$

The discriminant is given by: $\Delta=\left(e_1j_1\left(d_2+g\right)+e_3j_3\left(d_2+f\right)\right)^2-4e_1e_3j_1j_3d_2\left(f+g+d_2\right)$ which can be rewritten as $\Delta=\left(e_1j_1\left(d_2+g\right)-e_3j_3\left(d_2+f\right)\right)^2+4e_1e_3j_1j_3fg$. Since $4e_1e_3j_1j_3fg>0$, it follows that $\Delta>0$, resulting in two positive roots. To ensure the feasibility of each equilibrium point, it is imperative that both $S_2^*>0$ and $V_2^*>0$. Therefore, we have:

$$S_1^*=\frac{j_1(d_2+g)e_1+e_3j_3(d_2+f)\pm\sqrt{\Delta}}{2e_1e_3j_1j_3}$$

$$S_2^*=\frac{a(S_1{}^*)^2e_3j_3+(-Kae_3j_3+Kd_1e_3j_3-ag-ad_2)S_1^*+Kag+Kad_2-Kgd_1-Kd_1d_2}{K\left(-S_1^*e_1e_3j_3+fe_3+ge_1+d_2e_1\right)}$$

$$V_2^*=\frac{f\left(aK-d_1K-aS_1^*\right)}{K\left(-S_1^*e_1e_3j_3+fe_3+ge_1+d_2e_1\right)}$$

This equilibrium point is feasible when S_1^*, S_2^* and V_2^* are non-negative.

In the case when $S_1^*=\frac{\varepsilon+d_2+\iota k_2}{e_4j_4}$ (4.16) reduces to:

$$\begin{aligned}
&A_1S_2^*+B_1V_2^*+C_1T_2^*+D_1=0\\
&A_2S_2^*+gV_2^*+\varepsilon T_2^*=0\\
&fS_2^*+B_3V_2^*=0
\end{aligned}\tag{4.18}$$

where:

$$
\begin{aligned}
A_1 =& -e_1 j_4 e_4 K \\
A_2 =& \frac{j_1 (\imath k_2 + \varepsilon + d_2) e_1}{j_4 e_4} - f - d_2 \\
B_1 =& -K e_3 e_4 j_4 \\
B_3 =& \frac{e_3 j_3 (\imath k_2 + \varepsilon + d_2)}{j_4 e_4} - g - d_2 \\
C_1 =& -{e_4}^2 j_4 K \\
D_1 =& K a e_4 j_4 - d_1 j_4 e_4 K - a \imath k_2 - a\varepsilon - a d_2
\end{aligned}
$$

The disease-free equilibrium point is thus given by $(S_1^*, 0, S_2^*, 0, V_2^*, T_2^*)$ where:

$S_1^* = \frac{\varepsilon + d_2 + \imath k_2}{e_4 j_4}$

$S_2^* = \frac{B_3 \varepsilon D_1}{\varepsilon f B_1 - \varepsilon A_1 B_3 - f g C_1 + A_2 B_3 C_1}$

$V_2^* = -\frac{\varepsilon f D_1}{\varepsilon f B_1 - \varepsilon A_1 B_3 - f g C_1 + A_2 B_3 C_1}$

$T_2^* = \frac{(fg - A_2 B_3) D_1}{\varepsilon f B_1 - \varepsilon A_1 B_3 - f g C_1 + A_2 B_3 C_1}$

The corresponding equilibrium point is feasible when either of the following conditions is met:

(i) $D_1 < 0,\ B_3 < 0$ and $A_2 B_3 > fg$ when $\varepsilon f B_1 - \varepsilon A_1 B_3 - f g C_1 + A_2 B_3 C_1 > 0$

(ii) $D_1 > 0,\ B_3 > 0$ and $A_2 B_3 < fg$ when $\varepsilon f B_1 - \varepsilon A_1 B_3 - f g C_1 + A_2 B_3 C_1 < 0$

For the predator-free equilibrium point S_2^*, I_2^*, V_2^* and T_2^* are zero. It therefore follows:

$$
\begin{aligned}
a S_1^* \left(1 - \frac{S_1^*}{K}\right) - \beta_1 S_1^* I_1^* - d_1 S_1^* = 0 \\
\beta_1 S_1^* I_1^* - (d_1 + k_1) I_1^* = 0
\end{aligned}
\tag{4.19}
$$

The predator-free equilibrium point is given by $(S_1^*, I_1^*, 0, 0, 0, 0)$ where $S_1^* = \frac{d_1 + k_1}{\beta_1}$ and $I_1^* = \frac{K a \beta_1 - K \beta_1 d_1 - a d_1 - a k_1}{K {\beta_1}^2}$, which is feasible when $\frac{d_1}{a} + \frac{d_1 + k_1}{K \beta_1} < 1$.

With intervention in the predator only we examine the possibility that the prey go to extinction that is when $S_1^* = I_1^* = 0$, In this case we have:

$$\begin{aligned}
&-\beta_2 S_2 I_2 - f S_2 + g V_2 + \varepsilon T_2 - d_2 S_2 = 0 \\
&\beta_2 S_2 I_2 - h I_2 - (d_2 + k_2) I_2 = 0 \\
&f S_2 - g V_2 - d_2 V_2 = 0 \\
&h I_2 - \varepsilon T_2 - (d_2 + \imath k_2) T_2 = 0
\end{aligned} \tag{4.20}$$

The prey free equilibrium point is thus given by $(0, 0, S_2^*, I_2^*, V_2^*, T_2^*)$ where:

$$\begin{aligned}
S_2^* = &\quad \frac{h+d_2+k_2}{\beta_2} \\
I_2^* = &\quad -\frac{d_2\left(fh+fd_2+fk_2+gh+gd_2+gk_2+hd_2+{d_2}^2+d_2k_2\right)(\imath k_2+\varepsilon+d_2)}{\beta_2\left(gh\imath k_2+g\imath d_2k_2+g\imath {k_2}^2+h\imath d_2k_2+\imath {d_2}^2k_2+\imath d_2{k_2}^2+\varepsilon gd_2+\varepsilon gk_2+\varepsilon {d_2}^2+\varepsilon d_2k_2+ghd_2+g{d_2}^2+gd_2k_2+h{d_2}^2+{d_2}^3+{d_2}^2k_2\right)} \\
T_2^* = &\quad -\frac{hd_2\left(fh+fd_2+fk_2+gh+gd_2+gk_2+hd_2+{d_2}^2+d_2k_2\right)}{\beta_2\left(gh\imath k_2+g\imath d_2k_2+g\imath {k_2}^2+h\imath d_2k_2+\imath {d_2}^2k_2+\imath d_2{k_2}^2+\varepsilon gd_2+\varepsilon gk_2+\varepsilon {d_2}^2+\varepsilon d_2k_2+ghd_2+g{d_2}^2+gd_2k_2+h{d_2}^2+{d_2}^3+{d_2}^2k_2\right)} \\
V_2^* = &\quad \frac{f(h+d_2+k_2)}{\beta_2(g+d_2)}
\end{aligned}$$

This equilibrium point is not feasible since $I_2^*, T_2^* < 0$.

We will analyse three equilibrium points:

1. Trivial equilibrium point $(S_1, I_1, S_2, I_2, V_2, T_2) = (0, 0, 0, 0, 0, 0)$
2. Predator-free equilibrium point $(S_1, I_1, S_2, I_2, V_2, T_2) = (S_1^*, I_1^*, 0, 0, 0, 0)$
3. Disease-free equilibrium point $(S_1, I_1, S_2, I_2, V_2, T_2) = (S_1^*, 0, S_2^*, 0, V_2^*, 0)$

The Jacobian matrix is given as follows:

$$\begin{bmatrix} J_{11} & -\beta_1 S_1 & -S_1 e_1 & -S_1 e_2 & -S_1 e_3 & -e_4 S_1 \\ \beta_1 I_1 & J_{22} & -I_1 e_5 & -I_1 e_6 & -I_1 e_7 & -e_8 I_1 \\ j_1 S_2 e_1 & j_1 S_2 e_5 & J_{33} & -\beta_2 S_2 & g & \varepsilon \\ I_2 j_2 e_2 & I_2 j_2 e_6 & I_2 \beta_2 & J_{44} & 0 & 0 \\ V_2 j_3 e_3 & V_2 j_3 e_7 & f & 0 & J_{55} & 0 \\ e_4 T_2 j_4 & e_8 T_2 j_4 & 0 & h & 0 & J_{66} \end{bmatrix}$$

The variables J_{ij} used in the Jacobian matrix's description is given below:

Table 4.15: Model 3 description of elements, Jacobian matrix for model with human intervention in the predator only

Variable	**Description of variables**
J_{11}	$a\left(1-\frac{S_1}{K}\right)-\frac{aS_1}{K}-\beta_1 I_1-e_1 S_2-e_2 I_2-e_3 V_2-e_4 T_2-d_1$
J_{22}	$-e_6 I_2+\beta_1 S_1-e_5 S_2-e_8 T_2-e_7 V_2-d_1-k_1$
J_{33}	$j_1\left(I_1 e_5+S_1 e_1\right)-I_2\beta_2-f-d_2$
J_{44}	$j_2\left(I_1 e_6+S_1 e_2\right)+\beta_2 S_2-h-d_2-k_2$
J_{55}	$\left(I_1 e_7+S_1 e_3\right) j_3-g-d_2$
J_{66}	$-\varepsilon+\left(e_8 I_1+e_4 S_1\right) j_4-\imath k_2-d_2$

1. Trivial Equilibrium point $S_1 = I_2 = S_2 = I_2 = V_2 = T_2 = 0$

   The Jacobian matrix evaluated at the trivial equilibrium point $(S_1, I_1, S_2, I_2, V_2, T_2) = (0,0,0,0,0,0)$ gives the following:

$$\begin{bmatrix} a-d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_1-k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f-d_2 & 0 & g & \varepsilon \\ 0 & 0 & 0 & -h-d_2-k_2 & 0 & 0 \\ 0 & 0 & f & 0 & -g-d_2 & 0 \\ 0 & 0 & 0 & h & 0 & -\imath k_2-\varepsilon-d_2 \end{bmatrix}$$

The eigenvalues of the matrix are:

(i) $\lambda_1 = -d_1 - k_1$

(ii) $\lambda_2 = -h - d_2 - k_2$

(iii) $\lambda_3 = -f - g - d_2$

(iv) $\lambda_4 = -d_2$

(v) $\lambda_5 = -\imath k_2 - \varepsilon - d_2$

(vi) $\lambda_6 = a - d_1$

Since all the parameters are positive then $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 < 0$ and $\lambda_6 < 0$ when $a < d_1$. The trivial (mutual extinction) equilibrium point is stable if and only if $a < d_1$.

2. The predator-free equilibrium point

The Jacobian matrix evaluated at the predator-free equilibrium point $(S_1^*, I_1^*, 0, 0, 0, 0) = \left(\frac{d_1+k_1}{\beta_1}, \frac{Ka\beta_1 - K\beta_1 d_1 - ad_1 - ak_1}{K{\beta_1}^2}, 0, 0, 0, 0\right)$ is given below:

$$\begin{bmatrix} M_{11} & -d_1-k_1 & -\frac{e_1(d_1+k_1)}{\beta_1} & -\frac{e_2(d_1+k_1)}{\beta_1} & -\frac{e_3(d_1+k_1)}{\beta_1} & -\frac{e_4(d_1+k_1)}{\beta_1} \\ M_{21} & 0 & M_{23} & M_{24} & M_{25} & M_{26} \\ 0 & 0 & M_{33} & 0 & g & \varepsilon \\ 0 & 0 & 0 & M_{44} & 0 & 0 \\ 0 & 0 & f & 0 & M_{55} & 0 \\ 0 & 0 & 0 & h & 0 & M_{66} \end{bmatrix}$$

The variables M_{ij} used in the Jacobian matrix evaluated at the predator-free equilibrium point's description is given below:

Table 4.16: Model 3 description of elements, Jacobian matrix for model with human intervention in the predator only, at the predator-free equilibrium point

Variables	Description of variables
M_{11}	$-\frac{a(d_1+k_1)}{K\beta_1}$
M_{21}	$\frac{K(a-d_1)\beta_1-a(d_1+k_1)}{K\beta_1}$
M_{23}	$-\frac{(K(a-d_1)\beta_1-a(d_1+k_1))e_5}{K{\beta_1}^2}$
M_{24}	$-\frac{(K(a-d_1)\beta_1-a(d_1+k_1))e_6}{K{\beta_1}^2}$
M_{25}	$-\frac{(K(a-d_1)\beta_1-a(d_1+k_1))e_7}{K{\beta_1}^2}$
M_{26}	$-\frac{(K(a-d_1)\beta_1-a(d_1+k_1))e_8}{K{\beta_1}^2}$
M_{33}	$\frac{-K(f+d_2){\beta_1}^2+j_1((a-d_1)e_5+e_1(d_1+k_1))K\beta_1-ae_5j_1(d_1+k_1)}{K{\beta_1}^2}$
M_{44}	$\frac{-K(k_2+h+d_2){\beta_1}^2+j_2((a-d_1)e_6+e_2(d_1+k_1))K\beta_1-ae_6j_2(d_1+k_1)}{K{\beta_1}^2}$
M_{55}	$\frac{-K(g+d_2){\beta_1}^2+j_3((a-d_1)e_7+e_3(d_1+k_1))K\beta_1-ae_7j_3(d_1+k_1)}{K{\beta_1}^2}$
M_{66}	$\frac{-K(\imath k_2+\varepsilon+d_2){\beta_1}^2+Kj_4((a-d_1)e_8+e_4(d_1+k_1))\beta_1-ae_8j_4(d_1+k_1)}{K{\beta_1}^2}$

Two eigenvalues can be extracted from the Jacobian matrix namely:

(i) $\lambda_1 = \frac{-K(k_2+h+d_2){\beta_1}^2+j_2((a-d_1)e_6+e_2(d_1+k_1))K\beta_1-ae_6j_2(d_1+k_1)}{K{\beta_1}^2}$

(ii) $\lambda_2 = \frac{-K(\imath k_2+\varepsilon+d_2){\beta_1}^2+Kj_4((a-d_1)e_8+e_4(d_1+k_1))\beta_1-ae_8j_4(d_1+k_1)}{K{\beta_1}^2}$

since all the parameters are positive $\lambda_1, \lambda_2 < 0$ when $j_2\left((a-d_1)e_6+e_2(d_1+k_1)\right)K\beta_1 < K(k_2+h+d_2){\beta_1}^2 + ae_6j_2(d_1+k_1)$ and $K\left((a-d_1)e_8+e_4(d_1+k_1)\right)j_4\beta_1 < ae_8j_4(d_1+k_1) + K(\imath k_2+\varepsilon+d_2){\beta_1}^2$.

The reduced characteristic polynomial is given as the product of two quadratic polynomials:

$$(\lambda^2-(M_{33}+M_{55})\lambda+M_{33}M_{55}-fg)(\lambda^2-\lambda M_{11}+M_{21}d_1+M_{21}k_1)=0 \tag{4.21}$$

The respective Routh arrays are given as follows:

$$\begin{bmatrix} 1 & -fg+M_{33}M_{55} & \lambda^2 \\ -M_{33}-M_{55} & 0 & \lambda \\ -fg+M_{33}M_{55} & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & M_{21}d_1+M_{21}k_1 & \lambda^2 \\ -M_{11} & 0 & \lambda \\ M_{21}d_1+M_{21}k_1 & 0 & 1 \end{bmatrix}$$

The equilibrium point is stable when the eigenvalues $\lambda_1, \lambda_2 < 0$, $M_{33}+M_{55} < 0$, $M_{33}M_{55} > fg$ and $M_{21} > 0$.

3. Disease-free equilibrium point

   The Jacobian matrix evaluated at the disease-free equilibrium point $(S_1, I_1, S_2, I_2, V_2, T_2) = (S_1^*, 0, S_2^*, 0, V_1^*, 0)$ gives the following:

$$\begin{bmatrix} K_{11} & -\beta_1 S_1^* & -S_1^* e_1 & -S_1^* e_2 & -S_1^* e_3 & -e_4 S_1^* \\ 0 & K_{22} & 0 & 0 & 0 & 0 \\ j_1 S_2^* e_1 & j_1 S_2^* e_5 & K_{33} & -\beta_2 S_2^* & g & \varepsilon \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ V_2^* j_3 e_3 & V_2^* j_3 e_7 & f & 0 & K_{55} & 0 \\ 0 & 0 & 0 & h & 0 & K_{66} \end{bmatrix}$$

The variables K_{ij} used in the Jacobian matrix evaluated at the disease-free equilibrium point's description is given below:

Table 4.18: Model 3 description of elements, Jacobian matrix for the model with human intervention in the predator only at the disease-free equilibrium point

Variables	**Description of variables**
K_{11}	$a\left(1-\frac{S_1^*}{K}\right)-\frac{aS_1^*}{K}-e_1S_2^*-e_3V_2^*-d_1$
K_{22}	$\beta_1S_1^*-e_5S_2^*-e_7V_2^*-d_1-k_1$
K_{33}	$j_1S_1^*e_1-f-d_2$
K_{44}	$j_2S_1^*e_2+\beta_2S_2^*-h-d_2-k_2$
K_{55}	$S_1^*e_3j_3-g-d_2$
K_{66}	$e_4S_1^*j_4-\iota k_2-\varepsilon-d_2$

Three eigenvalues can be extracted from the Jacobian matrix namely:

(i) $\lambda_1=\beta_1S_1^*-e_5S_2^*-e_7V_2^*-d_1-k_1$

(ii) $\lambda_2=j_2S_1^*e_2+\beta_2S_2^*-h-d_2-k_2$

(iii) $\lambda_3=e_4S_1^*j_4-\iota k_2-\varepsilon-d_2$

In assessing the stability of the equilibrium point,

$\lambda_1<0$ when $\beta_1S_1^*<e_5S_2^*+e_7V_2^*+d_1+k_1$

$\lambda_2 < 0$ when $j_2 S_1^* e_2 + \beta_2 S_2^* < h + d_2 + k_2$

$\lambda_3 < 0$ when $e_4 S_1^* j_4 < \iota k_2 + \varepsilon + d_2$

The reduced characteristic polynomial is given by:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{4.22}$$

where:

$$\begin{aligned}
A = \quad & -\tfrac{1}{K}\left(KS_1^* e_1 j_1 + KS_1^* e_3 j_3 - KS_2^* e_1 - KV_2^* e_3 + aK - Kf - Kg - d_1 K - 2Kd_2 - 2aS_1^*\right) \\
B = \quad & -\tfrac{1}{K}(-K{S_1^*}^2 e_1 e_3 j_1 j_3 + KS_1^* S_2^* e_1 e_3 j_3 + KS_1^* V_2^* e_1 e_3 j_1 - aKS_1^* e_1 j_1 - KaS_1^* e_3 j_3 + \\
& KfS_1^* e_3 j_3 + KgS_1^* e_1 j_1 + KS_1^* d_1 e_1 j_1 + KS_1^* d_1 e_3 j_3 + KS_1^* d_2 e_1 j_1 + KS_1^* d_2 e_3 j_3 + \\
& 2a{S_1^*}^2 e_1 j_1 + 2a{S_1^*}^2 e_3 j_3 - KfS_2^* e_1 - KfV_2^* e_3 - KgS_2^* e_1 - KgV_2^* e_3 - 2KS_2^* d_2 e_1 - \\
& 2KV_2^* d_2 e_3 + aKf + Kag + 2Kad_2 - Kfd_1 - Kfd_2 - Kgd_1 - Kgd_2 - 2Kd_1 d_2 - K{d_2}^2 - \\
& 2afS_1^* - 2agS_1^* - 4aS_1^* d_2) \\
C = \quad & -\tfrac{1}{K}(KgS_1^* V_2^* e_1 e_3 j_1 - KgS_1^* V_2^* e_1 e_3 j_3 + KS_1^* S_2^* d_2 e_1 e_3 j_3 + KS_1^* V_2^* d_2 e_1 e_3 j_1 + \\
& aK{S_1^*}^2 e_1 e_3 j_1 j_3 - K{S_1^*}^2 d_1 e_1 e_3 j_1 j_3 - KfS_1^* S_2^* e_1 e_3 j_1 + KfS_1^* S_2^* e_1 e_3 j_3 - 2a{S_1^*}^3 e_1 e_3 j_1 j_3 - \\
& aKfS_1^* e_3 j_3 - aKgS_1^* e_1 j_1 - aKS_1^* d_2 e_1 j_1 - aKS_1^* d_2 e_3 j_3 + KfS_1^* d_1 e_3 j_3 + KgS_1^* d_1 e_1 j_1 + \\
& KS_1^* d_1 d_2 e_1 j_1 + KS_1^* d_1 d_2 e_3 j_3 + -KgV_2^* d_2 e_3 + 2af{S_1^*}^2 e_3 j_3 + 2ag{S_1^*}^2 e_1 j_1 + 2a{S_1^*}^2 d_2 e_1 j_1 + \\
& 2a{S_1^*}^2 d_2 e_3 j_3 - KfS_2^* d_2 e_1 - KfV_2^* d_2 e_3 - KgS_2^* d_2 e_1 - Kgd_1 d_2 - 2afS_1^* d_2 - 2agS_1^* d_2 - \\
& KS_2^* {d_2}^2 e_1 - KV_2^* {d_2}^2 e_3 + aKfd_2 + aKgd_2 - Kfd_1 d_2 + aK{d_2}^2 - d_1 K{d_2}^2 - 2aS_1^* {d_2}^2)
\end{aligned}$$

The Routh array is given by:

$$\begin{bmatrix} 1 & B & \lambda^3 \\ A & C & \lambda^2 \\ \frac{AB-C}{A} & 0 & \lambda \\ C & 0 & 1 \end{bmatrix}$$

The stability of the disease-free equilibrium point requires that all four conditions be met: the eigenvalues λ_1, λ_2 and λ_3 must be negative, and in addition $A > 0$, $C > 0$ and $AB > C$.

4.4 Model with human intervention in both species

4.4.1 Formulation of the model

In this model, treatment and vaccination are given to both species. The model is based on the following assumptions:

1. The prey population grows logistically. In the absence of preys, the population of predators will die out. The presence of the predator species has a negative impact on the growth of the prey.
2. Only the healthy population can reproduce.
3. There is disease in both species and infection can be transmitted from prey to predator through the process of predation.
4. In both populations, the susceptible population becomes infected by coming into contact with the infected population.
5. Both populations are divided into 4 classes: susceptible prey and predator (S_1 and S_2 respectively), infected prey and predators (I_1 and I_2 respectively) vaccinated prey and predators (V_1 and V_2 respectively) and preys and predators under treatment (T_1 and T_2 respectively).
6. The infected species never recover without treatment.
7. There is a natural death rate in all classes as well as a disease induced death rate in both infected populations.
8. Only the susceptible species are vaccinated which provides partial immunity, and only the infected species receives treatment.

9. After successful treatment, the species can still be infected at a later stage.
10. The infected prey is more likely to be caught compared to the healthy ones, and likewise infected predators are less efficient at catching prey than the healthy predators.

The following model can then be derived:

$$
\begin{aligned}
S_1' &= \underbrace{aS_1\left(1-\frac{S_1}{K}\right)}_{\text{growth}} \underbrace{-\beta_1 S_1 I_1}_{\text{infection}} \underbrace{-e_1S_1S_2 - e_2S_1I_2 - e_3S_1V_2 - e_4S_1T_2}_{\text{predation}} \underbrace{+\varepsilon_1 T_1}_{\text{treated}} \underbrace{+g_1V_1 - f_1S_1}_{\text{vaccination}} \underbrace{-d_1S_1}_{\text{mortality}} \\
I_1' &= \underbrace{\beta_1S_1I_1}_{\text{infection}} \underbrace{-e_5I_1S_2 - e_6I_1I_2 - e_7I_1V_2 - e_8I_1T_2}_{\text{predation}} \underbrace{-(d_1+k_1)I_1}_{\text{mortality}} \underbrace{-h_1I_1}_{\text{treatment}} \\
V_1' &= \underbrace{f_1S_1 - g_1V_1}_{\text{vaccination}} \underbrace{-e_9V_1S_2 - e_{10}V_1I_2 - e_{11}V_2V_1 - e_{12}T_2V_1}_{\text{predation}} \underbrace{-d_1V_1}_{\text{mortality}} \\
T_1' &= \underbrace{h_1I_1 - \varepsilon_1T_1}_{\text{treatment}} \underbrace{-e_{13}T_1S_2 - e_{14}T_1I_2 - e_{15}V_2T_1 - e_{16}T_2T_1}_{\text{predation}} \underbrace{-(d_1+i_1k_1)T_1}_{\text{mortality}} \\
S_2' &= \underbrace{(e_1S_1+e_5I_1+e_9V_1+e_{13}T_1)j_1S_2}_{\text{predation}} \underbrace{-\beta_2S_2I_2}_{\text{infection}} \underbrace{-f_2S_2+g_2V_2}_{\text{vaccination}} \underbrace{-d_2S_2}_{\text{mortality}} +\varepsilon_2T_2 \\
I_2' &= \underbrace{(e_2S_1+e_6I_1+e_{10}V_1+e_{14}T_1)j_2I_2}_{\text{predation}} \underbrace{+\beta_2S_2I_2}_{\text{infection}} \underbrace{-h_2I_2}_{\text{treatment}} \underbrace{-(d_2+k_2)I_2}_{\text{mortality}} \\
V_2' &= \underbrace{(e_3S_1+e_7I_1+e_{11}V_1+e_{15}T_1)j_3V_2}_{\text{predation}} \underbrace{+f_2S_2-g_2V_2}_{\text{vaccination}} \underbrace{-d_2V_2}_{\text{mortality}} \\
T_2' &= \underbrace{(e_4S_1+e_8I_1+e_{12}V_1+e_{16}T_1)j_4T_2}_{\text{predation}} \underbrace{+h_2I_2-\varepsilon_2T_2}_{\text{treatment}} \underbrace{-(d_2+i_2k_2)T_2}_{\text{mortality}}
\end{aligned}
\tag{4.23}
$$

The parameters' description is given below:

Table 4.20: Model 4 description of parameters

Parameter	**Description**
a	Intrinsic growth rate for prey
$\beta_{1,2}$	Infection rate in the prey and predator respectively

e_i	Predation rate with, $e_{susceptible} < e_{infected}$ for the same predator.
$d_{1,2}$	Natural death rate of the prey and predator respectively
$\varepsilon_{1,2}$	Recovery rate of the treated prey and predator respectively
$f_{1,2}$	Vaccination rate of the prey and predator respectively
$g_{1,2}$	Rate at which vaccine wears off in the prey and predator respectively
$h_{1,2}$	Treatment rate of the prey and predator respectively
$i_{1,2}$	Factor of disease induced death rate of the prey and predator respectively
$j_{1,2,3,4}$	Conversion rate of predation for susceptible, infected, vaccinated and treated predators respectively
K	Carrying capacity of the prey
$k_{1,2}$	Disease induced death rate for prey and predators respectively

The flowchart of the above model is given below in figure 4.4:

4.4.2 Qualitative analysis of the model

4.4.2.1 Positivity of solutions

In this section we show that under the given conditions all the solutions are positive, following a similar proof to that of [4].

Theorem 12 (Positivity)**.** *Let* $S_1(0) \geq 0$, $I_1(0) \geq 0$, $V_1(0) \geq 0$, $T_1(0) \geq 0$ $S_2(0) \geq 0$, $I_2(0) \geq 0$, $V_2(0) \geq 0$, $T_2(0) \geq 0$. *The solutions of (4.23) are non-negative for all* $t \geq 0$.

Proof. This will be shown in cases by contradiction.

Case 1. Let t_1 be such that $S_1(t_1) = 0$ and suppose that $S_1^{'}(t_1) < 0$ with $I_1(t) > 0$, $V_1(t) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_1$. In this case, substituting these

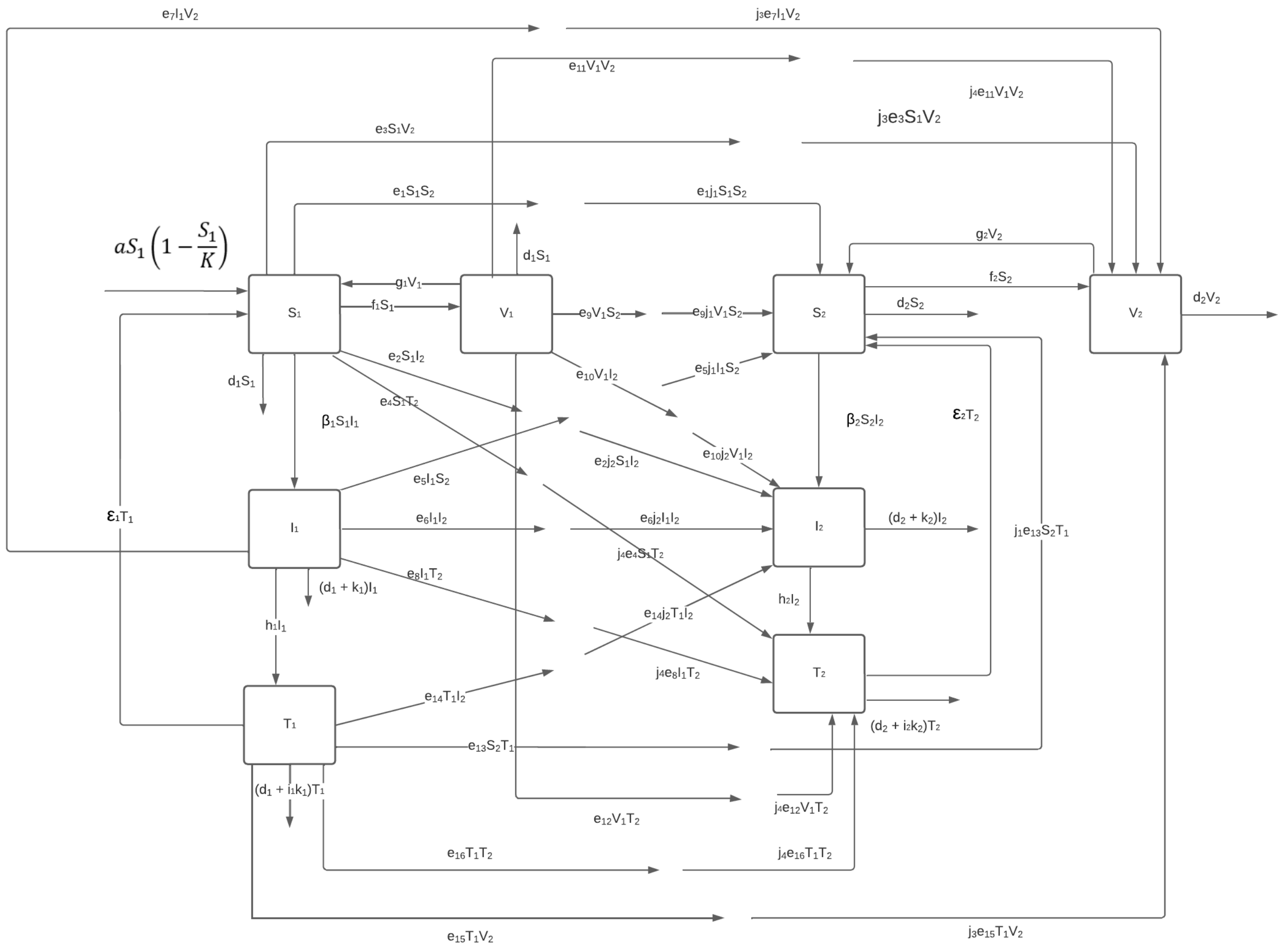

Figure 4.4: Flowchart of model 4 with human intervention on both species

values into (4.23) we get $S_1^{'}(t_1) = \varepsilon_1 T_1(t_1) > 0$, which is a contradiction to the assumption that $S_1^{'}(t_1) < 0$.

Case 2. Let t_2 be such that $I_1(t_2) = 0$ and suppose that $I_1^{'}(t_2) < 0$ with $S_1(t) > 0$, $V_1(t) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_2$. In this case, substituting these values into (4.23) we get $I_1^{'}(t_2) = 0$, which is a contradiction to the assumption that $I_1^{'}(t_2) < 0$.

Case 3. Let t_3 be such that $V_1(t_3) = 0$ and suppose that $V_1^{'}(t_3) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $T_1(t_1) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_3$. In this case, substituting these values into (4.23) we get $V_1^{'}(t_3) = f_1 S_1(t_3) > 0$, which is a contradiction to the assumption that $V_1^{'}(t_3) < 0$.

Case 4. Let t_4 be such that $T_1(t_4) = 0$ and suppose that $T_1^{'}(t_4) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $V_1(t_1) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_4$. In this case, substituting these values into (4.23) we get $T_1^{'}(t_4) = h_1 I_1(t_4) > 0$, which is a contradiction to the assumption that $T_1^{'}(t_4) < 0$.

Case 5. Let t_5 be such that $S_2(t_5) = 0$ and suppose that $S_2^{'}(t_5) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $V_1(t_1) > 0$, $T_1(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_5$. In this case, substituting these values into (4.23) we get $S_2^{'}(t_5) = g_2 V_1(t_5) > 0$, which is a contradiction to the assumption that $S_2^{'}(t_5) < 0$.

Case 6. Let t_6 be such that $I_2(t_6) = 0$ and suppose that $I_2^{'}(t_6) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $V_1(t_1) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $V_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_6$. In this case, substituting these values into (4.23) we get $I_2^{'}(t_6) = 0$, which is a contradiction to the assumption that $I_2^{'}(t_6) < 0$.

Case 7. Let t_7 be such that $V_2(t_7) = 0$ and suppose that $V_2^{'}(t_7) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $V_1(t_1) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $T_2(t) > 0$ for $0 < t < t_7$. In this case, substituting these values into (4.23) we get $V_2^{'}(t_7) = f_2 S_2(t_7) > 0$, which is a contradiction to the assumption that $V_2^{'}(t_7) < 0$.

Case 8. Let t_8 be such that $T_2(t_8) = 0$ and suppose that $T_2^{'}(t_8) < 0$ with $S_1(t) > 0$, $I_1(t) > 0$, $V_1(t_1) > 0$, $T_1(t) > 0$, $S_2(t) > 0$, $I_2(t) > 0$, $V_2(t) > 0$ for $0 < t < t_8$. In this case, substituting these values into (4.23) we get $T_2^{'}(t_8) = h_2 I_2(t_8) > 0$, which is a contradiction to the assumption that $T_2^{'}(t_8) < 0$.

Hence, all the solutions of (4.23) are non-negative. $\square$

Theorem 13 (Boundedness)**.** *Assume that $j_1, j_2, j_3, j_4 < 1$, then all solutions of the system (4.23) which initiate in $\mathbb{R}^8_+ = \{(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2),\ S_1 \geq 0,\ I_1 \geq 0,\ V_1 \geq 0,\ T_1 \geq 0,\ S_2 \geq 0,\ I_2 \geq 0,\ V_2 \geq 0,\ T_2 \geq 0\}$ are bounded.*

Proof. Let $N = S_1 + I_1 + V_1 + T_1 + S_2 + I_2 + V_2 + T_2$ then $\frac{dN}{dt} = S_1' + I_1' + V_1' + T_1' + S_2' + I_2' + V_2' + T_2'$.

$$\begin{aligned}
\frac{dN}{dt} = \;& aS_1\left(1 - \tfrac{S_1}{k}\right) + (j_1 - 1)\, e_1 S_1 S_2 + (j_2 - 1) e_2 S_1 I_2 + (j_3 - 1) e_3 S_1 V_2 + (j_4 - 1) e_4 S_1 T_2 - d_2 V_2 \\
& + (j_1 - 1)\, e_5 I_1 S_2 + (j_2 - 1) e_6 I_1 I_2 + (j_3 - 1) e_7 I_1 V_2 + (j_4 - 1) e_8 I_1 T_2 - d_1 S_1 - (d_1 + k_1) I_1 - d_2 S_2 \\
& + (j_1 - 1)\, e_9 V_1 S_2 + (j_2 - 1) e_{10} V_1 I_2 + (j_3 - 1) e_{11} V_1 V_2 + (j_4 - 1) e_{12} V_1 T_2 - d_1 V_1 - (d_1 + i_1 k_1) T_1 \\
& + (j_1 - 1)\, e_{13} T_1 S_2 + (j_2 - 1) e_{14} T_1 I_2 + (j_3 - 1) e_{15} T_1 V_2 + (j_4 - 1) e_{16} T_1 T_2 - (d_2 + k_2) I_2 - (d_2 + i_2 k_2) T_2
\end{aligned}$$

since $j_1, j_2, j_3, j_4 < 1$ we have $(j_1 - 1) < 0$, $(j_2 - 1) < 0$, $(j_3 - 1) < 0$ and $(j_4 - 1) < 0$. Therefore:

$$\begin{aligned}
\frac{dN}{dt} \leq \;& aS_1\left(1 - \tfrac{S_1}{k}\right) - d_1 S_1 - (d_1 + k_1) I_1 - d_1 V_1 - (d_1 + i_1 k_1) T_1 - d_2 V_2 - (d_2 + i_2 k_2) T_2 - d_2 S_2 - (d_2 + k_2) I_2 \\
\leq \;& aS_1 - \tfrac{aS_1^2}{k} - d_1 S_1 - d_1 V_1 - (d_1 + i_1 k_1) T_1 - (d_1 + k_1) I_1 - d_2 V_2 - (d_2 + i_2 k_2) T_2 - d_2 S_2 - (d_2 + k_2) I_2 \\
\leq \;& aS_1 - (d_1 S_1 + d_1 V_1 + (d_1 + i_1 k_1) T_1 + (d_1 + k_1) I_1 + d_2 V_2 + (d_2 + i_2 k_2) T_2 + d_2 S_2 + (d_2 + k_2) I_2) \\
\leq \;& \mu - \eta N
\end{aligned}$$

where: $\mu = a\alpha$, $\alpha = \max\{S_1(0), k\}$ and $\eta = \min\{d_1, d_2\}$.

We then have: $\frac{dN}{dt} + \eta N \leq \alpha$ which is a linear first order differential equation. The integration factor is given by $e^{\int \eta dt} = e^{\eta t}$.

$e^{\eta t} \frac{dN}{dt} + \eta e^{\eta t} N \leq \alpha e^{\eta t}$

$\frac{d}{dt}\left(N e^{\eta t}\right) \leq \alpha e^{\eta t}$

$N e^{\eta t} \leq \frac{\alpha}{\eta} e^{\eta t} + c$

Given the initial condition $N(0) = N_0 \geq 0$, we have $N \leq \frac{\alpha}{\eta} + \left(N_0 + \frac{\alpha}{\eta}\right) e^{-\eta t}$.

As $t \to \infty$, the exponential term $e^{-\eta t} \to 0$. Consequently, we can deduce that $N \leq \frac{\alpha}{\eta}$. Hence all the solutions of (4.23) are bounded and are confined to the region $\Omega = \left\{(S_1, I_1, S_2, I_2, V_2, T_2) \in \mathbb{R}_+^8 : N \leq \frac{\alpha}{\eta}\right\}$.

□

4.4.2.2 Stability of the equilibrium points

The equilibrium points of the system are obtained by setting the right-hand side of (4.23) to zero. When all the components $(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2)$ are zero, the equations remain valid, and we obtain the trivial equilibrium point (0,0,0,0,0,0,0,0). Moreover, for the disease-free equilibrium, we require I_1^* and I_2^* to be zero. It therefore follows that:

$$\begin{aligned}
&aS_1^*\left(1-\frac{S_1^*}{K}\right)-e_1S_1^*S_2^*-e_3S_1^*V_2^*+g_1V_1^*-f_1S_1^*-d_1S_1^*-e_4S_1^*T_2^*+\varepsilon_1T_1^*=0\\
&-S_2^*V_1^*e_9-V_1^*V_2^*e_{11}+f_1S_1^*-V_1^*d_1-g_1V_1^*-e_{12}T_2^*V_1^*=0\\
&-\varepsilon_1T_1^*-e_{13}T_1^*S_2^*-e_{15}V_2^*T_1^*-e_{16}T_2^*T_1^*-(d_1+i_1k_1)T_1^*=0\\
&j_1S_2^*\left(e_1S_1^*+e_9V_1^*+e_{13}T_1^*\right)-f_2S_2^*+g_2V_2^*-d_2S_2^*+\varepsilon_2T_2^*=0\\
&j_3V_2^*\left(e_3S_1^*+e_{11}V_1^*+e_{15}T_1^*\right)+f_2S_2^*-g_2V_2^*-d_2V_2^*=0\\
&(e_4S_1^*+e_{12}V_1^*+e_{16}T_1^*)j_4T_2^*-\varepsilon_2T_2^*-(d_2+i_2k_2)T_2^*=0
\end{aligned}\tag{4.24}$$

From the third equation of (4.24), we can deduce that either $T_1^* = 0$ or $-\varepsilon_1 - e_{13}S_2^* - e_{15}V_2^* - e_{16}T_2^* - d_1 - i_1k_1 = 0$. Since all parameters, S_2^*, V_2^*, and T_2^*, are positive, we can conclude that $T_1^* = 0$. Similarly, from the last equation of (4.24), it follows that either $T_2^* = 0$ or $(e_4S_1^* + e_{12}V_1^*)j_4 - \varepsilon_2 - (d_2 + i_2k_2) = 0$. Considering the former condition, (4.24) reduces to:

$$
\begin{aligned}
& aS_1^*\left(1-\frac{S_1^*}{K}\right)-e_1S_1^*S_2^*-e_3S_1^*V_2^*+g_1V_1^*-f_1S_1^*-d_1S_1^*=0\\
& -S_2^*V_1^*e_9-V_1^*V_2^*e_{11}+f_1S_1^*-V_1^*d_1-g_1V_1^*=0\\
& j_1S_2^*\left(e_1S_1^*+e_9V_1^*\right)-f_2S_2^*+g_2V_2^*-d_2S_2^*=0\\
& j_3V_2^*\left(e_3S_1^*+e_{11}V_1^*\right)+f_2S_2^*-g_2V_2^*-d_2V_2^*=0
\end{aligned}
\tag{4.25}
$$

The disease-free equilibrium point is then given by $(S_1,I_1,V_1,T_1,S_2,I_2,V_2,T_2) = (S_1^*,0,V_1^*,0,S_2^*,0,V_2^*,0)$ where S_1^* is a positive root, when it exists, of the equation ${AS_1}^2+BS_1+C=0$ where:

$$
\begin{aligned}
A = \quad & A_1A_2-\tfrac{A_1B_2e_4}{e_{12}}-\tfrac{B_1e_4A_2}{e_{12}}+\tfrac{B_1B_2{e_4}^2}{{e_{12}}^2}\\
B = \quad & \tfrac{A_1B_2\iota k_2}{j_4e_{12}}-2\tfrac{B_1e_4B_2d_2}{{e_{12}}^2j_4}-2\tfrac{B_1e_4B_2\varepsilon_2}{{e_{12}}^2j_4}+\tfrac{B_1\iota k_2A_2}{j_4e_{12}}-2\tfrac{B_1e_4B_2\iota k_2}{{e_{12}}^2j_4}+\tfrac{A_1B_2d_2}{j_4e_{12}}+\tfrac{A_1B_2\varepsilon_2}{j_4e_{12}}+\tfrac{B_1d_2A_2}{j_4e_{12}}+\\
& \tfrac{B_1\varepsilon_2A_2}{j_4e_{12}}+A_1C_2+C_1A_2-\tfrac{B_1e_4C_2}{e_{12}}-\tfrac{C_1B_2e_4}{e_{12}}\\
C = \quad & -1+2\tfrac{B_1\iota k_2B_2d_2}{{e_{12}}^2{j_4}^2}+2\tfrac{B_1\iota k_2B_2\varepsilon_2}{{e_{12}}^2{j_4}^2}+\tfrac{B_1\iota^2{k_2}^2B_2}{{e_{12}}^2{j_4}^2}+\tfrac{B_1\iota k_2C_2}{j_4e_{12}}+2\tfrac{B_1d_2B_2\varepsilon_2}{{e_{12}}^2{j_4}^2}+\tfrac{C_1B_2\iota k_2}{j_4e_{12}}+\tfrac{B_1{d_2}^2B_2}{{e_{12}}^2{j_4}^2}+\\
& \tfrac{B_1d_2C_2}{j_4e_{12}}+\tfrac{B_1{\varepsilon_2}^2B_2}{{e_{12}}^2{j_4}^2}+\tfrac{B_1\varepsilon_2C_2}{j_4e_{12}}+\tfrac{C_1B_2d_2}{j_4e_{12}}+\tfrac{C_1B_2\varepsilon_2}{j_4e_{12}}+C_1C_2
\end{aligned}
$$

The variables used in the derivation of this disease-free equilibrium point are described below:

Table 4.23: Model 4 description variables used in the disease-free equilibrium point

Variable	**Description of variables**
A_1	$-\frac{e_3j_3}{f_2}$
A_2	$-\frac{e_1j_1}{g_2}$
B_1	$-\frac{e_{11}j_3}{f_2}$
B_2	$-\frac{e_9j_1}{g_2}$
C_1	$\frac{g_2+d_2}{f_2}$
C_2	$\frac{f_2+d_2}{g_2}$

$V_1^* = \frac{-S_1^*e_4j_4+i_2k_2+d_2+\varepsilon_2}{j_4e_{12}}$

$$V_2^* = -\frac{\big(\big(S_1^*(e_1e_{12}-e_4e_9)j_1-e_{12}(d_2+f_2)\big)j_4+j_1e_9(i_2k_2+d_2+\varepsilon_2)\big)\big(((d_1+g_1)e_4+e_{12}f_1)S_1^*j_4-(d_1+g_1)(i_2k_2+d_2+\varepsilon_2)\big)}{\big(\big(j_1e_{11}(e_1e_{12}-e_4e_9)S_1^*-e_{12}(g_2e_9+e_{11}(d_2+f_2))\big)j_4+j_1e_{11}e_9(i_2k_2+d_2+\varepsilon_2)\big)\big(S_1^*e_4j_4-i_2k_2-d_2-\varepsilon_2\big)}$$

$$S_2^* = \frac{e_{12}\big(((d_1+g_1)e_4+e_{12}f_1)S_1^*j_4-(d_1+g_1)(i_2k_2+d_2+\varepsilon_2)\big)j_4g_2}{\big(\big(\big(\big(S_1^*e_1j_1-d_2-f_2\big)e_{12}-S_1^*e_4e_9j_1\big)e_{11}-e_9e_{12}g_2\big)j_4+j_1e_{11}e_9(i_2k_2+d_2+\varepsilon_2)\big)\big(S_1^*e_4j_4-i_2k_2-d_2-\varepsilon_2\big)}$$

The equilibrium point is feasible when $S_1^*, V_1^*, S_2^*, V_2^* > 0$.

In the case, when we consider the latter condition, that is $(e_4S_1+e_{12}V_1)j_4-\varepsilon_2-(d_2+i_2k_2)=0$. (4.24) can be rewritten as:

$$\begin{aligned}
&A_5S_1^*S_2^*+B_5S_1^*T_2^*+C_5S_1^*V_2^*+D_5S_1^*+E_5V_1^*+F_5{S_1^*}^2=0\\
&A_4S_2^*V_1^*+B_4V_1^*T_2^*+C_4V_1^*V_2^*+D_4V_1^*+E_4S_1^*=0\\
&A_3S_1^*S_2^*+B_3S_2^*V_1^*+C_3S_2^*+D_3V_2^*+E_3T_2^*=0\\
&A_1S_1^*V_2^*+B_1V_1^*V_2^*+C_1S_2^*+D_1V_2^*=0\\
&A_2S_1^*+B_2V_1^*+C_2=0
\end{aligned}\tag{4.26}$$

The variables used in the derivation of this disease-free equilibrium point are described below:

Table 4.24: Model 4 description variables used in the disease-free equilibrium point when $T_2 \neq 0$

Variable	Description	Variable	Description	Variable	Description
A_1	e_3j_3	B_1	$e_{11}j_3$	C_1	f_2
A_2	e_4j_4	B_2	$e_{12}j_4$	C_2	$-i_2k_2-d_2-\varepsilon_2$
A_3	e_1j_1	B_3	e_9j_1	C_3	$-d_2-f_2$
A_4	$-e_9$	B_4	$-e_{12}$	C_4	$-e_{11}$
A_5	$-Ke_1$	B_5	$-Ke_4$	C_5	$-Ke_3$
D_1	$-d_2-g_2$	D_3	g_2	D_4	$-d_1-g_1$
D_5	$Ka-Kd_1-Kf_1$	E_3	ε_2	E_4	f_1

E_5	Kg_1	F_5	$-a$		

This disease-free equilibrium point is then given by $(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2) = (S_1^*, 0, V_1^*, 0, S_2^*, 0, V_2^*, T_2^*)$ where S_1^* is a positive root, when it exists, of the equation $AS_1^{*2} + BS_1^* + C = 0$

$$\begin{aligned}
V_1^* &= -\tfrac{\alpha_{11}}{B_2}\\
S_2^* &= -\tfrac{\alpha_{66}\alpha_{44}}{\alpha_{11}\alpha_{22}}\\
T_2^* &= -\tfrac{\alpha_{33}\alpha_{44}}{\alpha_{11}\alpha_{22}}\\
V_2^* &= \tfrac{\alpha_{55}\alpha_{44}}{\alpha_{11}\alpha_{22}}
\end{aligned}$$

Where

$$\begin{aligned}
\alpha_{11} = {}& A_2S_1^* + C_2\\
\alpha_{22} = {}& A_1A_2B_2B_3B_4S_1^{*2} - A_1A_3{B_2}^2B_4S_1^{*2} - {A_2}^2B_1B_3B_4S_1^{*2} + A_2A_3B_1B_2B_4S_1^{*2} + A_1A_4{B_2}^2E_3S_1^* -\\
& A_1{B_2}^2B_4C_3S_1^* + A_1B_2B_3B_4C_2S_1^* - A_2A_4B_1B_2E_3S_1^* + A_2B_1B_2B_4C_3S_1^* - 2A_2B_1B_3B_4C_2S_1^* +\\
& A_2B_2B_3B_4D_1S_1^* + A_3B_1B_2B_4C_2S_1^* - A_3{B_2}^2B_4D_1S_1^* - A_4B_1B_2C_2E_3 + A_4{B_2}^2D_1E_3 +\\
& B_1B_2B_4C_2C_3 - B_1B_3B_4{C_2}^2 + {B_2}^2B_4C_1D_3 - {B_2}^2B_4C_3D_1 - {B_2}^2C_1C_4E_3 + B_2B_3B_4C_2D_1\\
\alpha_{33} = {}& A_1A_2B_2B_3S_1^{*2} - A_1A_3{B_2}^2S_1^{*2} - {A_2}^2B_1B_3S_1^{*2} + A_2A_3B_1B_2S_1^{*2} - A_1{B_2}^2C_3S_1^* +\\
& A_1B_2B_3C_2S_1^* + A_2B_1B_2C_3S_1^* - 2A_2B_1B_3C_2S_1^* + A_2B_2B_3D_1S_1^* + A_3B_1B_2C_2S_1^* -\\
& A_3{B_2}^2D_1S_1^* + B_1B_2C_2C_3 - B_1B_3{C_2}^2 + {B_2}^2C_1D_3 - {B_2}^2C_3D_1 + B_2B_3C_2D_1\\
\alpha_{44} = {}& A_2D_4S_1^* - E_4S_1^*B_2 + C_2D_4\\
\alpha_{55} = {}& {B_2}^2C_1E_3\\
\alpha_{66} = {}& (A_1B_2S_1^* - A_2B_1S_1^* - B_1C_2 + B_2D_1)\,B_2E_3
\end{aligned}$$

The equilibrium point is feasible when $S_1^*, V_1^*, S_2^*, V_2^*, T_2^* > 0$.

We will analyse the two equilibrium points:

1. Trivial equilibrium point $(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2) = (0,0,0,0,0,0,0,0)$

2. Disease-free equilibrium point $(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2) = (S_1^*, 0, V_1^*, 0, S_2^*, 0, V_2^*, 0)$

The Jacobian matrix is given as follows:

$$\begin{bmatrix}
J_{11} & -\beta_1 S_1 & g_1 & \varepsilon_1 & -e_1 S_1 & -e_2 S_1 & -e_3 S_1 & -e_4 S_1 \\
\beta_1 I_1 & J_{22} & 0 & 0 & -e_5 I_1 & -e_6 I_1 & -e_7 I_1 & -e_8 I_1 \\
f_1 & 0 & J_{33} & 0 & -e_9 V_1 & -e_{10} V_1 & -e_{11} V_1 & -e_{12} V_1 \\
0 & h_1 & 0 & J_{44} & -e_{13} T_1 & -e_{14} T_1 & -e_{15} T_1 & -e_{16} T_1 \\
j_1 S_2 e_1 & j_1 S_2 e_5 & j_1 S_2 e_9 & j_1 S_2 e_{13} & J_{55} & -\beta_2 S_2 & g_2 & \varepsilon_2 \\
j_2 I_2 e_2 & j_2 I_2 e_6 & j_2 I_2 e_{10} & j_2 I_2 e_{14} & \beta_2 I_2 & J_{66} & 0 & 0 \\
j_3 V_2 e_3 & j_3 V_2 e_7 & j_3 V_2 e_{11} & j_3 V_2 e_{15} & f_2 & 0 & J_{77} & 0 \\
j_4 T_2 e_4 & j_4 T_2 e_8 & j_4 T_2 e_{12} & j_4 T_2 e_{16} & 0 & h_2 & 0 & J_{88}
\end{bmatrix}$$

The variables J_{ij} used in the Jacobian matrix's description is given below:

Table 4.27: Model 4 description of elements of the Jacobian matrix of the model with human intervention in both species

Variable	**Description of variables**
J_{11}	$a\left(1 - \frac{S_1}{K}\right) - \frac{aS_1}{K} - \beta_1 I_1 - e_1 S_2 - e_2 I_2 - e_3 V_2 - e_4 T_2 - f_1 - d_1$
J_{22}	$-e_6 I_2 + \beta_1 S_1 - e_5 S_2 - e_8 T_2 - e_7 V_2 - d_1 - h_1 - k_1$
J_{33}	$-I_2 e_{10} - S_2 e_9 - T_2 e_{12} - V_2 e_{11} - d_1 - g_1$
J_{44}	$-I_2 e_{14} - S_2 e_{13} - T_2 e_{16} - V_2 e_{15} - i_1 k_1 - d_1 - \varepsilon_1$
J_{55}	$j_1 (e_5 I_1 + e_1 S_1 + e_{13} T_1 + e_9 V_1) - \beta_2 I_2 - f_2 - d_2$
J_{66}	$j_2 (e_6 I_1 + e_2 S_1 + e_{14} T_1 + V_1 e_{10}) + \beta_2 S_2 - h_2 - d_2 - k_2$
J_{77}	$j_3 (e_7 I_1 + e_3 S_1 + e_{15} T_1 + e_{11} V_1) - g_2 - d_2$
J_{88}	$j_4 (e_8 I_1 + e_4 S_1 + e_{16} T_1 + e_{12} V_1) - \varepsilon_2 - i_2 k_2 - d_2$

We then analyse two equilibrium points to assess their stability:

1. Trivial Equilibrium point

   The Jacobian matrix evaluated at the trivial equilibrium point $(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2) = (0,0,0,0,0,0,0,0)$ gives the following:

$$\begin{bmatrix} a-f_1-d_1 & 0 & g_1 & \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & -d_1-h_1-k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_1 & 0 & -d_1-g_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & -i_1k_1-d_1-\varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_2-d_2 & 0 & g_2 & \varepsilon_2 \\ 0 & 0 & 0 & 0 & 0 & -h_2-d_2-k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_2 & 0 & -g_2-d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_2 & 0 & -i_2k_2-d_2-\varepsilon_2 \end{bmatrix}$$

   Four eigenvalues can be extracted from the Jacobian matrix namely:

   (i) $\lambda_1 = -h_2 - d_2 - k_2$

   (ii) $\lambda_2 = -i_2k_2 - d_2 - \varepsilon_2$

   (iii) $\lambda_3 = -d_1 - h_1 - k_1$

   (iv) $\lambda_4 = -i_1k_1 - d_1 - \varepsilon_1$

   Since all the parameters are positive then $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$.

   The reduced characteristic polynomial is given by:

$$(\lambda + d_2)(\lambda + d_2 + f_2 + g_2)\left(-\lambda^2 + (a - 2d_1 - f_1 - g_1)\lambda + ad_1 + ag_1 - {d_1}^2 - d_1f_1 - d_1g_1\right) = 0 \tag{4.27}$$

   From which two more eigenvalues can be extracted:

(i) $\lambda_5 = -d_2$

(ii) $\lambda_6 = -d_2 - f_2 - g_2$

Since all the parameters are positive, $\lambda_5, \lambda_6 < 0$.

The following quadratic polynomial remains:

$\lambda^2 - (a - 2d_1 - f_1 - g_1)\lambda - ad_1 - ag_1 + {d_1}^2 + d_1 f_1 + d_1 g_1 = 0$

The Routh array is given by:

$$\begin{bmatrix} 1 & -ad_1 - ag_1 + {d_1}^2 + d_1 f_1 + d_1 g_1 & \lambda^2 \\ -a + 2d_1 + f_1 + g_1 & 0 & \lambda \\ -ad_1 - ag_1 + {d_1}^2 + d_1 f_1 + d_1 g_1 & 0 & 1 \end{bmatrix}$$

Since the eigenvalues λ_1 to λ_6 are negative, the trivial equilibrium point will be stable when $a < 2d_1 + f_1 + g_1$ and $d_1 + f_1 + g_1 > a(1 + \frac{g_1}{d_1})$

2. Disease-free equilibrium point

The Jacobian matrix evaluated at the disease-free equilibrium point $(S_1, I_1, V_1, T_1, S_2, I_2, V_2, T_2) = (S_1^*, 0, V_1^*, 0, S_2^*, 0, V_2^*, 0)$ gives the following:

$$\begin{bmatrix} K_{11} & -\beta_1 S_1^* & g_1 & \varepsilon_1 & -e_1 S_1^* & -e_2 S_1^* & -e_3 S_1^* & -e_4 S_1^* \\ 0 & K_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ f_1 & 0 & K_{33} & 0 & -e_9 V_1^* & -e_{10} V_1^* & -e_{11} V_1^* & -e_{12} V_1^* \\ 0 & h_1 & 0 & K_{44} & 0 & 0 & 0 & 0 \\ j_1 S_2^* e_1 & j_1 S_2^* e_5 & j_1 S_2^* e_9 & j_1 S_2^* e_{13} & K_{55} & -\beta_2 S_2^* & g_2 & \varepsilon_2 \\ 0 & 0 & 0 & 0 & 0 & K_{66} & 0 & 0 \\ j_3 V_2^* e_3 & j_3 V_2^* e_7 & j_3 V_2^* e_{11} & j_3 V_2^* e_{15} & f_2 & 0 & K_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & h_2 & 0 & K_{88} \end{bmatrix}$$

The variables K_{ij} used in the Jacobian matrix evaluated at the disease-free equilibrium point is explained below:

Table 4.28: Description of elements, Jacobian matrix evaluated at the disease-free equilibrium point of the model with human intervention in both species

Variable	**Description of variable**
K_{11}	$a\left(1-\frac{S_1^*}{K}\right)-\frac{aS_1^*}{K}-e_1S_2^*-e_3V_2^*-f_1-d_1$
K_{22}	$\beta_1S_1^*-e_5S_2^*-e_7V_2^*-d_1-h_1-k_1$
K_{33}	$-S_2^*e_9-V_2^*e_{11}-d_1-g_1$
K_{44}	$-e_{13}S_2^*-e_{15}V_2^*-i_1k_1-d_1-\varepsilon_1$
K_{55}	$j_1\left(e_1S_1^*+e_9V_1^*\right)-f_2-d_2$
K_{66}	$j_2\left(e_2S_1^*+e_{10}V_1^*\right)+\beta_2S_2^*-h_2-d_2-k_2$
K_{77}	$j_3\left(e_3S_1^*+e_{11}V_1^*\right)-g_2-d_2$
K_{88}	$j_4\left(e_4S_1^*+e_{12}V_1^*\right)-\varepsilon_2-i_2k_2-d_2$

Four eigenvalues can be extracted from the Jacobian matrix namely:

(i) $\lambda_1 = \beta_1 S_1^* - e_5 S_2^* - e_7 V_2^* - d_1 - h_1 - k_1$

(ii) $\lambda_2 = -e_{13} S_2^* - e_{15} V_2^* - i_1 k_1 - d_1 - \varepsilon_1$

(iii) $\lambda_3 = j_2 \left(e_2 S_1^* + e_{10} V_1^*\right) + \beta_2 S_2^* - h_2 - d_2 - k_2$

(iv) $\lambda_4 = j_4 \left(e_4 S_1^* + e_{12} V_1^*\right) - \varepsilon_2 - i_2 k_2 - d_2$

In assessing the stability of the equilibrium point,

$\lambda_1 < 0$ when $\beta_1 S_1^* < e_5 S_2^* + e_7 V_2^* + d_1 + h_1 + k_1$

$\lambda_2 < 0$ since all parameters as well as $S_2^*, V_2^* > 0$.

$\lambda_3 < 0$ when $j_2 \left(e_2 S_1^* + e_{10} V_1^*\right) + \beta_2 S_2^* < h_2 + d_2 + k_2$

$\lambda_4 < 0$ when $j_4 \left(e_4 S_1^* + e_{12} V_1^*\right) < \varepsilon_2 + i_2 k_2 + d_2$

The reduced characteristic polynomial is given by:

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0 \tag{4.28}$$

where:

$$\begin{aligned}
A = {} & -K_{11} - K_{33} - K_{55} - K_{77} \\
B = {} & j_1 S_2^* {e_1}^2 S_1^* + j_3 V_2^* {e_3}^2 S_1^* + j_1 S_2^* {e_9}^2 V_1^* + j_3 V_2^* {e_{11}}^2 V_1^* + K_{33} K_{11} + K_{55} K_{11} + K_{77} K_{11} + \\
& K_{55} K_{33} + K_{77} K_{33} + K_{77} K_{55} - f_1 g_1 - f_2 g_2 \\
C = {} & -K_{11} S_2^* V_1^* {e_9}^2 j_1 - K_{11} V_1^* V_2^* {e_{11}}^2 j_3 - K_{33} S_1^* S_2^* {e_1}^2 j_1 - K_{33} S_1^* V_2^* {e_3}^2 j_3 - K_{55} S_1^* V_2^* {e_3}^2 j_3 - \\
& K_{55} V_1^* V_2^* {e_{11}}^2 j_3 - K_{77} S_1^* S_2^* {e_1}^2 j_1 - K_{77} S_2^* V_1^* {e_9}^2 j_1 + S_1^* S_2^* e_1 e_3 f_2 j_1 + S_1^* S_2^* e_1 e_9 f_1 j_1 + \\
& S_1^* V_2^* e_1 e_3 g_2 j_3 + S_1^* V_2^* e_3 e_{11} f_1 j_3 + S_2^* V_1^* e_1 e_9 g_1 j_1 + S_2^* V_1^* e_9 e_{11} f_2 j_1 + V_1^* V_2^* e_3 e_{11} g_1 j_3 + \\
& V_1^* V_2^* e_9 e_{11} g_2 j_3 - K_{11} K_{33} K_{55} - K_{11} K_{33} K_{77} - K_{11} K_{55} K_{77} + K_{11} f_2 g_2 - K_{33} K_{55} K_{77} + \\
& K_{33} f_2 g_2 + K_{55} f_1 g_1 + K_{77} f_1 g_1
\end{aligned}$$

$$\begin{aligned} D \quad & S_1^*S_2^*V_1^*V_2^*{e_1}^2{e_{11}}^2 j_1 j_3 - 2S_1^*S_2^*V_1^*V_2^* e_1 e_3 e_9 e_{11} j_1 j_3 + S_1^*S_2^*V_1^*V_2^*{e_3}^2{e_9}^2 j_1 j_3 + \\ & K_{11}K_{55}V_1^*V_2^*{e_{11}}^2 j_3 + K_{11}K_{77}S_2^*V_1^*{e_9}^2 j_1 - K_{11}S_2^*V_1^* e_9 e_{11} f_2 j_1 - K_{11}V_1^*V_2^* e_9 e_{11} g_2 j_3 + \\ & K_{33}K_{55}S_1^*V_2^*{e_3}^2 j_3 + K_{33}K_{77}S_1^*S_2^*{e_1}^2 j_1 - K_{33}S_1^*S_2^* e_1 e_3 f_2 j_1 - K_{33}S_1^*V_2^* e_1 e_3 g_2 j_3 - \\ & K_{55}S_1^*V_2^* e_3 e_{11} f_1 j_3 - K_{55}V_1^*V_2^* e_3 e_{11} g_1 j_3 - K_{77}S_1^*S_2^* e_1 e_9 f_1 j_1 - K_{77}S_2^*V_1^* e_1 e_9 g_1 j_1 + \\ & S_1^*S_2^* e_3 e_9 f_1 f_2 j_1 + S_1^*V_2^* e_1 e_{11} f_1 g_2 j_3 + S_2^*V_1^* e_1 e_{11} f_2 g_1 j_1 + V_1^*V_2^* e_3 e_9 g_1 g_2 j_3 + \\ & K_{11}K_{33}K_{55}K_{77} - K_{11}K_{33} f_2 g_2 - K_{55}K_{77} f_1 g_1 + f_1 f_2 g_1 g_2 \end{aligned}$$

The Routh array is given by:

$$\begin{bmatrix} 1 & B & D & \lambda^4 \\ A & C & 0 & \lambda^3 \\ \frac{AB-C}{A} & D & 0 & \lambda^2 \\ \frac{ABC-A^2D-C^2}{AB-C} & 0 & 0 & \lambda \\ D & 0 & 0 & 1 \end{bmatrix}$$

The predator-free equilibrium point is stable when the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0, A > 0, AB > C$, $ABC > A^2D + C^2$ and $D > 0$.

4.5 Conclusion

Four models describing the effects of human intervention in the form of treatment and vaccination on a predator-prey system have been proposed and studied. The positivity of each model's solutions is established, and the dynamical behaviour of the system has been analytically investigated at the equilibrium points. To confirm the analytical results, models (4.1), (4.9), (4.15), and (4.23) are numerically solved in the following chapter.

Chapter 5

Quantitative Analysis

Numerical simulations involve the solving of differential equations that describe how a system's behavior changes over time. Numerical simulations are necessary to study the behavior of systems for which mathematical models are too complex to yield analytical solutions. In this chapter, we perform computer simulations, which are essential for validating the analytical results obtained in the chapter on qualitative analysis. We present numerical simulations of solutions for each of the four models and compare their respective figures.

5.1 Parameter selection

In this section, we look at how we choose the parameters. A model similar to (4.1) where there is no human intervention is found in [13] from which most of the parameters estimations are extracted. A few of the parameters that are not found in [13] are taken from [10]. A summary of the parameters used is given below:

Table 5.1: Parameters of the model without human intervention and their values

Symbol	Meaning/ Description	Value	Source
a	Intrinsic growth rate for prey	3	Hu and Li [10]

K	Carrying capacity of the prey	100	Kant and Kumar [13]
d_1	Natural death rate of the prey	0.125	Kant and Kumar [13]
d_2	Natural death rate of the predator	0.125	Kant and Kumar [13]
e_1	Predation rate of the susceptible predators on the healthy prey	0.25	Hu and Li [10]
e_2	Predation rate of the infected predators on the healthy prey	0.125	Kant and Kumar [13]
e_3	Predation rate of the susceptible predators on the infected prey	0.125	Kant and Kumar [13]
e_4	Predation rate of the infected predators on the infected prey	0.0625	Kant and Kumar [13]
j_1	Conversion rate of predation for the susceptible predators	0.125	Kant and Kumar [13]
j_2	Conversion rate of predation for the infected predators	0.0625	Kant and Kumar [13]
k_1	Disease induced death rate for the preys	0.25	Hu and Li [10]
k_2	Disease induced death rate for the predators	0.25	Kant and Kumar [13]
β_1	Infection rate in the prey	0.5	Kant and Kumar [13]
β_2	Infection rate in the predators	0.25	Kant and Kumar [13]

5.2 Numerical simulations

In this section, we solve the system of ODEs using the in-built ordinary differential equation solver MATLAB function ode45.

5.2.1 Model with no human intervention

In this model, without human intervention, it is observed that the populations of susceptible and infected prey oscillate for a while as the amplitude decreases before it eventually reaches an equi-

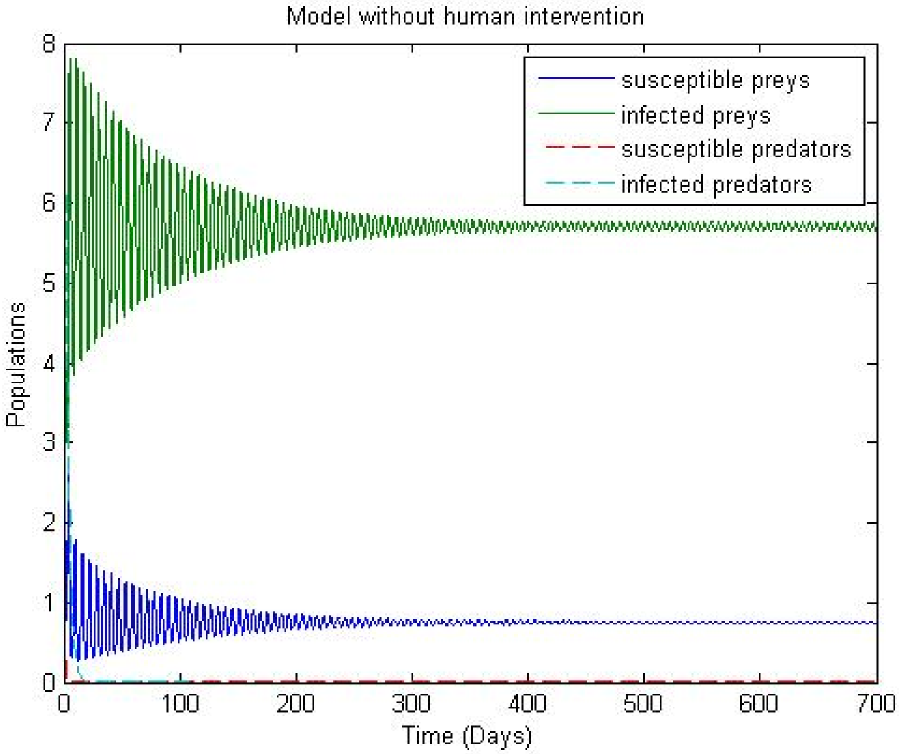

Figure 5.1: Variation in the populations: model without human intervention, with $S_1(0) = 5$, $I_1(0) = 5$, $S_2(0) = 5$ and $I_2(0) = 5$.

librium point while the predator populations, both susceptible and infected predators eventually die out.

With the above parameter values (see Table (5.1)) the predator-free equilibrium point which is given by $(S_1, I_1, S_2, I_2) = (S_1^*, I_1^*, 0, 0)$ where $S_1^* = 0.75$ and $I_1^* = 5.705$ is stable.

Indeed, the obvious eigenvalues are $\lambda_1 = -0.012421875$ and $\lambda_2 = -0.3468554688$ and they are both negative. The reduced characteristic polynomial is given as follows:

$$\lambda^2 + 0.02250000000\,\lambda + 1.069687500 = 0 \tag{5.1}$$

Since the obvious eigenvalues are proven to be negative and the reduced polynomial is quadratic of the form $\lambda^2 + a\lambda + b$ then according to the Routh-Hurwitz criterion since $a, b > 0$ then the equilibrium point is stable. This indeed confirms the theoretical findings.

5.2.2 Model with human intervention in the prey only

The values of parameters in model (4.9) with human intervention in the prey are presented below. Parameters that are the same as in model (4.1) without human intervention, such as the a : Intrinsic growth rate for prey, are carried over from Table (5.1). The parameter values for the vaccinated prey, e_5 and e_6 are chosen to be the same as those of the susceptible prey. The parameter values of the prey under treatment, e_7 and e_8 are chosen to be the same as those of the infected prey.

Table 5.3: Parameters of the model with human intervention in the prey only

Symbol	Meaning/ Description	Value	Source
e_1	Predation rate of the susceptible predators on the susceptible prey	0.25	Same as Table 5.1
e_2	Predation rate of the infected predators on the susceptible prey	0.125	Same as Table 5.1
e_3	Predation rate of the susceptible predators on the infected prey	0.125	Same as Table 5.1
e_4	Predation rate of the infected predators on the infected prey	0.0625	Same as Table 5.1
e_5	Predation rate of the susceptible predators on the vaccinated prey	0.25	(Assumed)
e_6	Predation rate of the infected predators on the vaccinated prey	0.125	(Assumed)
e_7	Predation rate of the susceptible predators on the prey under treatment	0.11	(Assumed)

e_8	Predation rate of the infected predators on the prey under treatment	0.05	(Assumed)
f	Vaccination rate of prey	0.2	Cojacaru, Migot and Jaber [5]
g	Rate at which the vaccine wears off	0.2	Cojacaru, Migot and Jaber [5]
h	Treatment rate of the infected prey	0.2	Cojacaru, Migot and Jaber [5]
i	Factor of disease induced death rate	0.2	Cojacaru, Migot and Jaber [5]
ε	Recovery rate of the treated prey	0.02	Cojacaru, Migot and Jaber [5]

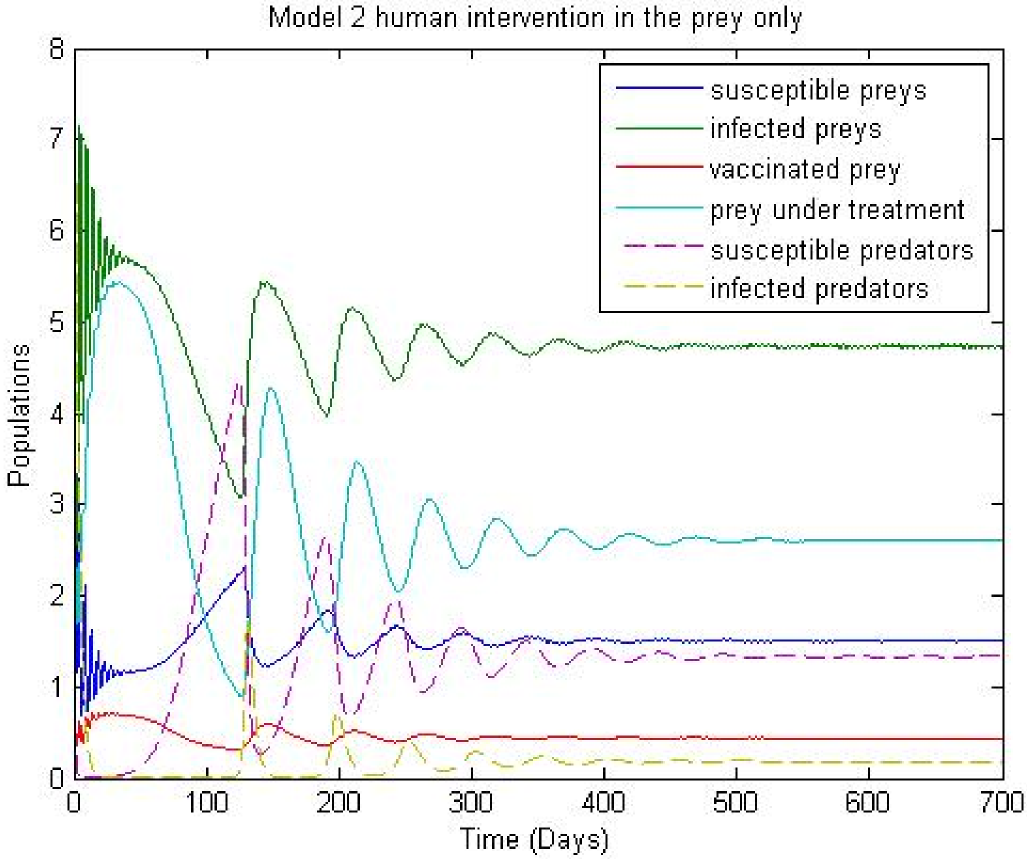

Figure 5.2: Variation in the populations: model with human intervention in the prey only, with $S_1(0) = 5$, $I_1(0) = 5$, $V_1(0) = 5$, $T_1(0) = 5$, $S_2(0) = 5$ and $I_2(0) = 5$.

In this model, using the given parameters we observe that introducing human intervention on the prey, has an overall positive effect as both species persist compared to model (4.1) where the predator dies out as time goes on.

The graph of the variation in the populations for the model with human intervention in the prey only shown, in figure (5.2) clearly suggests the existence of an equilibrium point that is neither the trivial nor the disease-free equilibrium point. This endemic equilibrium point which is given by $(S_1^*, I_1^*, V_1^*, T_1^*, S_2^*, I_2^*) = (1.505990427, 4.734375937, 0.4422941550, 2.606879272, 1.332555492, 0.1828124287)$ appears to be stable since when solutions begin near this equilibrium point, trajectories consistently move towards it as time progresses. This behavior strongly suggests stability.

The trivial equilibrium point given by $(S_1^*, I_1^*, V_1^*, T_1^*, S_2^*, I_2^*) = (0,0,0,0,0,0)$ is stable if and only if $a < f + g + 2d_1$ i.e. $3 < 0.65$ as well as $f + g + d_1 > \frac{ag}{d_1} + a$ i.e. $7.8 < 0.525$. It is sufficient to demonstrate that the failure of either of the inequalities to hold implies the instability of the equilibrium point. In this case, as both inequalities fail to hold, we can conclude that the trivial equilibrium point is unstable, thereby confirming the theoretical findings.

Similarly the disease-free equilibrium point is given by $(S_1^*, I_1^*, V_1^*, T_1^*, S_2^*, I_2^*)$ where $S_1^* = 3.742062289$, $S_2^* = 10.30609596$, $V_1^* = 0.2579377114$ and $I_1^* = T_1^* = I_2^* = 0$. The three eigenvalues that can be extracted from the Jacobian matrix are $\lambda_1 = 0.007769149$, $\lambda_2 = -1.341170556$ and $\lambda_3 = 2.232773990$. Since two of the eigenvalues are positive this equilibrium point is unstable confirming the theoretical findings.

The predator-free equilibrium point is given by $(S_1^*, I_1^*, V_1^*, T_1^*, 0, 0)$ where $S_1^* = 1.150000000$, $I_1^* = 5.718881676$, $V_1^* = 0.7076923077$, $T_1^* = 5.512175109$. The two eigenvalues extracted from the Jacobian matrix evaluated at this equilibrium points are $\lambda_1 = 0.949730285$ and $\lambda_2 = -0.3217292976$, since λ_1 is positive this shows that the equilibrium point is unstable. This also confirms the theoretical findings. Therefore, if the model has an equilibrium point, it does not belong to those studied in the chapter on qualitative analysis. The parameter values used in this simulation yield the graph shown in Figure (5.2), which suggests the existence of an equilibrium point where all the values are positive. This equilibrium point is indeed the coexistence equilibrium point with disease.

5.2.3 Model with human intervention in predator only

Like model (4.9) with human intervention in the prey the parameters given below are taken from [5], [10] and [13].

Table 5.5: Parameters of the model with human intervention in the predator only

Symbol	Meaning/ Description	Value	Source
e_1	Predation rate of the susceptible predators on the susceptible prey	0.25	Hu and Li [10]
e_2	Predation rate of the infected predators on the susceptible prey	0.125	Kant and Kumar [13]
e_3	Predation rate of the vaccinated predators on the susceptible prey	0.125	Kant and Kumar [13]
e_4	Predation rate of the predators under treatment on the susceptible prey	0.0625	Kant and Kumar [13]
e_5	Predation rate of the susceptible predators on the infected prey	0.25	(Assumed)
e_6	Predation rate of the infected predators on the infected prey	0.125	(Assumed)
e_7	Predation rate of the vaccinated predators on the infected prey	0.11	(Assumed)

e_8	Predation rate of the predators under treatment on the infected prey	0.05	(Assumed)
f	Vaccination rate of prey	0.2	Cojacaru, Migot and Jaber [5]
g	Rate at which the vaccine wears off	0.2	Cojacaru, Migot and Jaber [5]
h	Treatment rate of the infected prey	0.2	Cojacaru, Migot and Jaber [5]
i	Factor of disease induced death rate	0.2	Cojacaru, Migot and Jaber [5]
j_3	Conversion rate of predation for the vaccinated predators	0.125	(Assumed)
j_4	Conversion rate of predation for the predators under treatment	0.0625	(Assumed)
ε	Recovery rate of the treated prey	0.02	Cojacaru, Migot and Jaber [5]

With the given parameters it is observed that in the beginning both prey populations oscillate until they eventually go towards the equilibrium point, the predators however decline to extinction. This suggests that human intervention failed however a different set of parameters would lead to disease-free equilibrium point.

The predator-free equilibrium point is given by $(S_1^*, I_1^*, S_2^*, I_2^*, V_2^*, T_2^*)$ where $S_1^* = 0.75$ and $I_1^* = 5.705$ and $S_2^* = I_2^* = V_2^* = T_2^* = 0$. The eigenvalues from the Jacobian matrix are thus $\lambda_1 = -0.5468554688$ and $\lambda_2 = -0.1793554688$. The reduced characteristic polynomial is given as follows:

$$50.0\,\lambda^4 + 22.36718750\,\lambda^3 + 54.21847687\,\lambda^2 + 22.72826587\,\lambda + 0.2740032867 = 0 \qquad (5.2)$$

The corresponding Routh array is given below:

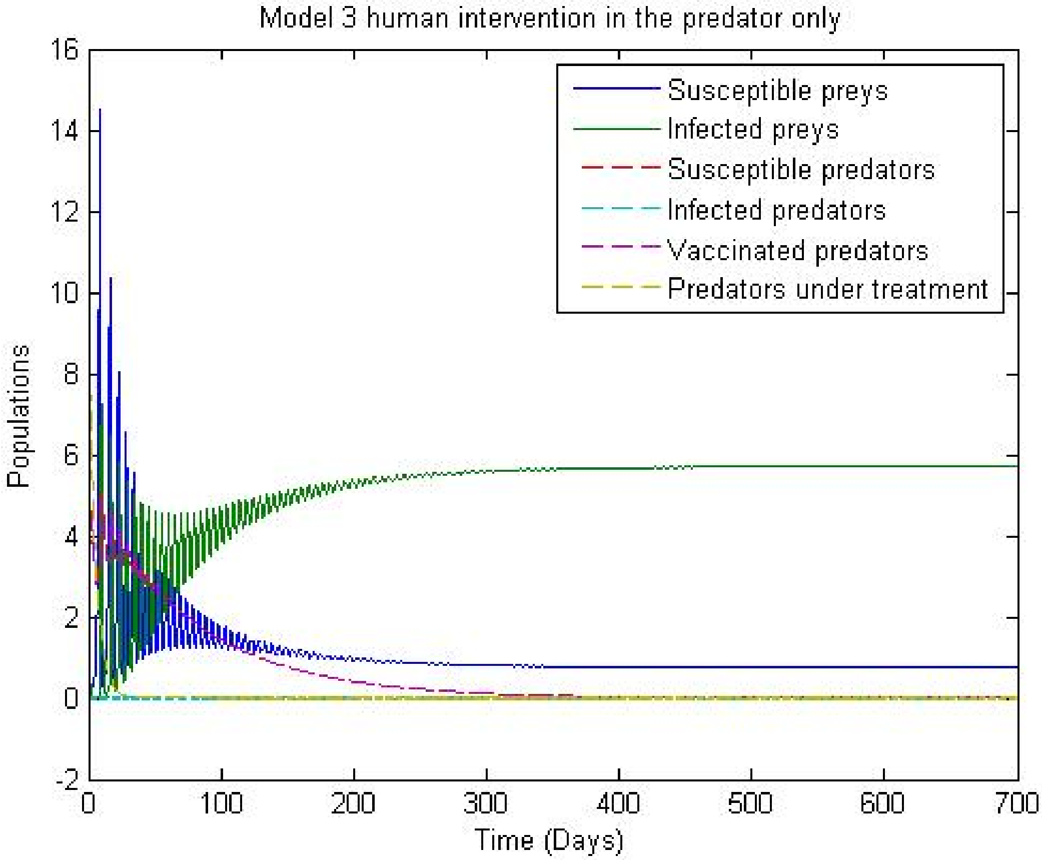

Figure 5.3: Variation in the populations: model with human intervention in the predator only, with $S_1(0) = 5$, $I_1(0) = 5$, $S_2(0) = 5$, $I_2(0) = 5$, $V_2(0) = 5$ and $T_2(0) = 5$.

$$\begin{bmatrix} 50.0 & 54.21847687 & 0.2740032867 & \lambda^4 \\ 22.36718750 & 22.72826587 & 0.0 & \lambda^3 \\ 3.411316001 & 0.2740032867 & 0.0 & \lambda^2 \\ 20.93169150 & 0.0 & 0.0 & \lambda \\ 0.2740032867 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

Since both eigenvalues are negative and all the elements in the first column of the array are positive we can conclude that the equilibrium point is stable. This indeed confirms the theoretical findings.

The trivial equilibrium point is unstable since its only eigenvalue that was not proved to be negative under qualitative analysis is positive, the given condition for negativity was that $a < d_1$ and since $a = 3$ and $d_1 = 0.125$ the condition is not met and the equilibrium point is unstable.

Similarly the disease-free equilibrium point is given by $(S_1^*, 0, S_2^*, 0, V_2^*, 0)$ where $S_1^* = 4$, $S_2^* = 5.51$, $V_2^* = 5.51$ and $I_1^* = T_1^* = I_2^* = 0$. The three eigenvalues that can be extracted from the Jacobian matrix are $\lambda_1 = -0.1762500$, $\lambda_2 = 0.8337500$ and $\lambda_3 = 0.24750$. Since two of the eigenvalues are positive this equilibrium point is unstable, confirming the theoretical findings.

It's worth noting that human intervention is carried out with the aim of preventing species extinction and, ideally, maintaining the species in a disease-free state. However, in this model, intervention in the predator population appears to lead to a predator-free equilibrium point, indicating that the intervention has not succeeded. One possible explanation for this undesired outcome can be attributed to the choice of parameter values. These parameters may unexpectedly influence the dynamics of predator-prey interactions, potentially favouring the prey species and suppressing the predators more than originally anticipated. These findings underscore the significance of thorough planning and assessment of intervention strategies to ensure the achievement of desired ecological outcomes.

5.2.4 Model with human intervention in both species

The values of parameters which are taken from [5] and [10] are given below.

Table 5.7: Parameters of the model with human intervention in both species

Symbol	Meaning/ Description	Value	Source
e_1	Predation rate of the susceptible predators on the susceptible prey	0.25	Same as Table 5.1
e_2	Predation rate of the infected predators	0.125	Same as Table 5.1

	on the susceptible prey		
e_3	Predation rate of the vaccinated predators on the susceptible prey	0.25	Same as Table 5.5
e_4	Predation rate of the predators under treatment on the susceptible prey	0.125	Same as Table 5.5
e_5	Predation rate of the susceptible predators on the infected prey	0.125	Same as Table 5.1
e_6	Predation rate of the infected predators on the infected prey	0.0625	Same as Table 5.1
e_7	Predation rate of the vaccinated predators on the infected prey	0.125	Same as Table 5.5
e_8	Predation rate of the predators under treatment on the infected prey	0.0625	Same as Table 5.5
e_9	Predation rate of the susceptible predators on the vaccinated prey	0.25	Same as Table 5.3
e_{10}	Predation rate of the infected predators on the vaccinated prey	0.125	Same as Table 5.3
e_{11}	Predation rate of the vaccinated predators on the vaccinated prey	0.25	(Assumed)
e_{12}	Predation rate of the predators under treatment on the vaccinated prey	0.125	(Assumed)
e_{13}	Predation rate of the susceptible predators on the prey under treatment	0.125	Same as Table 5.3
e_{14}	Predation rate of the infected predators on the prey under treatment	0.0625	Same as Table 5.3
e_{15}	Predation rate of the vaccinated predators	0.125	(Assumed)

	on the prey under treatment		
e_{16}	Predation rate of the predators under treatment on the prey under treatment	0.0625	(Assumed)
f_1	Vaccination rate of prey	0.2	Same as Table 5.3
f_2	Vaccination rate of predator	0.2	Same as Table 5.5
g_1	Rate at which the prey vaccine wears off	0.2	Same as Table 5.3
g_2	Rate at which the predator vaccine wears off	0.2	Same as Table 5.5
h_1	Treatment rate of the infected prey	0.2	(Assumed)
h_2	Treatment rate of the infected predator	0.2	Same as Table 5.5
i_1	Factor of disease induced death rate in the prey	0.25	Same as Table 5.3
i_2	Factor of disease induced death rate in the predator	0.25	Same as Table 5.5
ε_1	Recovery rate of the treated prey	0.02	Same as Table 5.3
ε_2	Recovery rate of the treated predator	0.02	Same as Table 5.5

In this model, it is observed that initially, the populations oscillate before eventually reaching an equilibrium point. Unlike the previous model with human intervention in the predator population, it can be seen that none of the populations declines to zero.

The graph of the variation in the populations for the model with human intervention in both species, in figure (5.4) clearly suggests the existence of an equilibrium point that is neither the trivial nor the disease-free, but a coexistence equilibrium point with disease (endemic) for which the numeric values are given by $S_1^* = 2.3085$, $I_1^* = 2.9615$,$V_1^* = 0.3112$, $T_1^* = 0.7528$, $S_2^* = 2.1601$, $I_2^* = 0.1296$, $V_2^* = 2.3340$, $T_2^* = 0.1502$

The trivial equilibrium point given by $(S_1^*, I_1^*, V_1^*, T_1^*, S_2^*, I_2^*, V_2^*, T_2^*) = (0,0,0,0,0,0,0,0)$. The eigenvalues that were extracted from the Jacobian matrix have already been shown to be negative

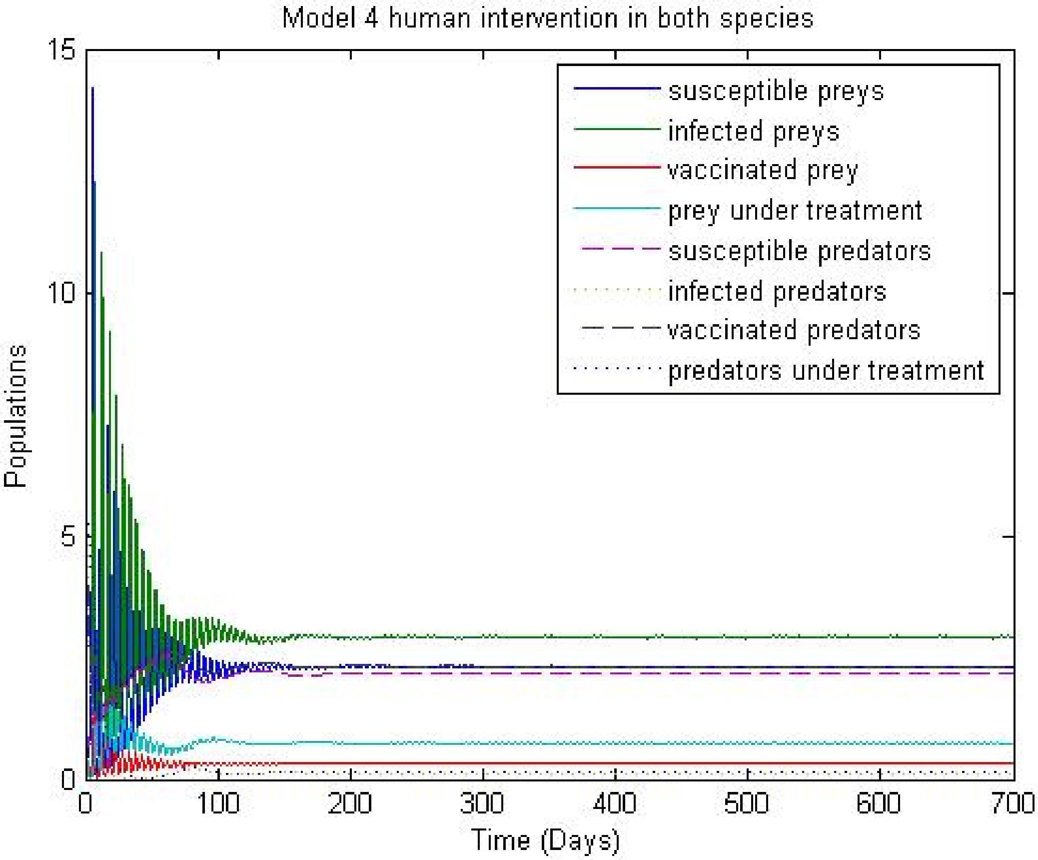

Figure 5.4: Variation in the populations: model with human intervention in both species with $S_1(0) = 5$, $I_1(0) = 5$, $V_1(0) = 5$, $T_1(0) = 5$ $S_2(0) = 5$, $I_2(0) = 5$, $V_2(0) = 5$ and $T_2(0) = 5$.

under qualitative analysis and we thus have a reduced quadratic characteristic polynomial.

$$\lambda^2 - 0.850\lambda - 0.421875 = 0 \tag{5.3}$$

According to the Routh-Hurwitz criterion for assessing the stability of a quadratic polynomial in the form of $\lambda^2 + a\lambda + b$, both conditions $a > 0$ and $b > 0$ must hold for stability. In the current scenario, we observe that $a < 0$ and $b < 0$, which implies that the criterion is not met, indicating the instability of the equilibrium point. This observation aligns with the theoretical findings and strengthens the argument for the existence of another equilibrium point.

5.3 Conclusion

In this chapter, numerical simulations were employed to validate the analytical results obtained in the previous chapter on qualitative analysis. MATLAB was utilized for conducting these simulations, and the outcomes were presented for each model. In the model with no human intervention, it was observed that the predator populations eventually declined to extinction, underscoring the necessity of human intervention. In contrast, in the model with human intervention in the prey population only, positive effects were observed on both prey and predator populations. The simulations suggested the existence of an endemic equilibrium point, which appeared to be stable. The stability of the calculated equilibrium points in all the models were assessed through two methods: eigenvalue analysis, which involved calculating eigenvalues from the Jacobian matrix evaluated at each equilibrium point and evaluating their real parts, and the Routh-Hurwitz criteria. Both approaches yielded results consistent with the theoretical findings.

Chapter 6

Discussion and Conclusion

In the pursuit of understanding the dynamics of a predator-prey system with infection in both species as well as the impact of human intervention, in this mini-thesis we formulated and analysed a series of models. These models shed light on the potential effectiveness of human interventions, namely treatment and vaccination, in preserving the populations of both predator and prey species. In this chapter, we provide discussions on the four formulated models and end the chapter with the conclusion and some recommendations.

6.1 Discussion

The initial model, presented in (4.1), offered insights into the system's behaviour in the absence of human intervention. The biological feasibility of the model was shown by establishing the positivity of solutions. All 16 equilibrium points ranging from mutual extinction to coexistence were found and their feasibility conditions established. Of particular interest were the mutual extinction, disease-free, predator-free and endemic equilibrium points. These equilibrium points were then analysed qualitatively. The stability of all equilibrium points that were of interest was conducted using factorization of the characteristic polynomial, that lead to a combination of the study of signs of the eigenvalues that could be extracted from the Jacobian matrix in an obvious manner, and the use of the Routh-Hurwitz criterion on the reduced characteristic polynomial. Stability conditions for all four equilibrium points were given. In the numerical analysis of the model, a scenario was shown where the predators faced extinction (predator-free equilibrium point) without human intervention, underscoring the necessity of such intervention.

Our exploration then expanded to incorporate human intervention in the prey population only, as indicated in (4.9) and similarly biological feasibility was shown by establishing the positivity of the solutions. Because of the complexity of the formulated model only two equilibrium points were assessed, the trivial and disease-free equilibrium points. Interestingly, the question of potential predator extinction arose, since the intervention focuses on the prey only. In an effort to answer this question, the predator-free equilibrium point was also analysed. Qualitative analysis of the three equilibrium points was done to provide the conditions under which they are stable. These stability conditions were found by using the eigenvalues of the Jacobian matrix as well as the Routh-Hurwitz criteria. Quantitative analysis confirmed that predators persist under the given conditions, reinforcing the beneficial role of prey-focused intervention and establishing that human intervention in the prey alone contributes positively to the entire system.

Turning our attention to the predator population, a third model (4.15) which included human intervention in the predators was formulated. Likewise, the positivity of the solutions was established. Stability analysis was centred around the trivial and disease-free equilibrium point employing eigenvalue assessments and the Routh-Hurwitz criteria, and the conditions for stability were given. Addressing the concern of prey extinction due to predator-focused intervention, we examined the feasibility of a prey-free equilibrium point. The prey-free equilibrium point was found and analysed. Conclusively, it was determined that extinction of the prey is implausible with predator-targeted human intervention because this equilibrium point was not feasible. In the quantitative analysis under the given conditions, it was observed that the predators decline to extinction even with the human intervention. This led to qualitative analysis of the predator-free equilibrium point by extracting the obvious eigenvalues from the characteristic polynomial and analysing the reduced characteristic polynomial using the Routh-Hurwitz criteria. The stability conditions were also given. The model showed that the intervention strategy did not work as expected, resulting in the extinction of the predators. This unexpected outcome might be due to the specific values chosen for certain parameters. These parameter values seem to have unintentionally shifted the balance in favour of the prey and reduced the predator population more than intended. These findings highlight the importance of carefully planning and evaluating intervention strategies to make sure they have the desired impact on the ecosystem.

The final model (4.23), encompassing treatment and vaccination in both species underwent positivity analysis of the solutions followed by the investigation of its equilibrium points. The trivial equilibrium point was analysed for stability using the eigenvalues of the Jacobian matrix. Quantitative analysis unveiled persistent coexistence of both species under the specified conditions, as well as the existence of an endemic equilibrium point. Numerical assessment deemed the disease-free equilibrium point unstable according to the Routh-Hurwitz criteria.

In summary, the models presented in this mini-thesis collectively demonstrate the profound influence of human intervention on the dynamics of predator-prey systems with infection. Treatment and vaccination emerge as potent tools to avert extinction and sustain coexistence. This study contributes valuable insights into the intricate interplay between ecological and epidemiological factors, paving the way for more informed strategies in disease management and wildlife conservation.

6.2 Conclusion

Our investigation underscores the vital role of human intervention in maintaining the predator-prey system amidst infection in both species. Notably, the objectives outlined in section 1.3 have been successfully met, with the mathematical model effectively describing the relationship between infection and human intervention. By meticulously analysing various equilibrium points and conducting numerical simulations, we have gained insights into the effectiveness of diverse intervention strategies for preventing the decline of both populations. Further research is imperative to enhance the practicality of these findings. The formulated models yield valuable perspectives into the intricate dynamics of predator-prey relationships, laying the groundwork for future research in ecological preservation and disease control.

6.3 Recommendations

The comprehensive analysis and findings presented in this mini-thesis open avenues for future research and practical applications in the realm of predator-prey systems and disease dynamics. Building upon the insights gained, the following recommendations are proposed. Further research could delve into optimizing the parameters associated with treatment and vaccination. One potential avenue is to consider that, due to the partial immunity provided by vaccination, enhancing models with vaccination may involve incorporating prey or predators transitioning from the vaccination class to the infected class at a rate that is less than those moving from the susceptible class to the infected class. By exploring a range of intervention intensities and timing, we can identify strategies that minimize disease prevalence while maintaining ecological balance. To enhance the practical relevance of the models, future studies should consider incorporating additional ecological complexities that may influence disease dynamics. Factors such as migration or competition interactions could be integrated into the models as well as the inclusion of predators acquiring the infection through predation would contribute to a more realistic representation of the predator-prey ecosystems. This expansion would enable a deeper exploration of the dynamic interplay between ecological and epidemiological processes. Regarding the managing of game parks, it is crucial to exercise caution, striking a balance between human intervention and the preservation of both predator and prey species to protect the fragile harmony of species within a game park. Evaluating the parameter values to grasp their impact on the ecosystem is key to making well-informed decisions.

In conclusion, the research undertaken in this mini-thesis lays a foundation for advancing our understanding of predator-prey systems with infection and underscores the importance of human intervention. By pursuing the outlined recommendations, researchers can contribute to the development of more nuanced and effective strategies for disease control and ecological preservation. These endeavours have the potential to yield practical insights with far-reaching implications.

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ETHICAL CLEARANCE CERTIFICATE

Ethical Clearance Reference Number: SOS-0070 Date: 02 June 2022

This Ethical Clearance Certificate is issued by the University of Namibia Ethics Committee (REC) in accordance with the University of Namibia's Research Ethics Policy and Guidelines. Ethical approval is given in respect of undertakings contained in the Research Project outlined below. This Certificate is issued on the recommendations of the ethical evaluation done by the ethics committee.

Title of Project:	ANALYSIS OF PREDATOR-PREY MODELS WITH INFECTION IN BOTH THE PREDATOR AND PREY AND HUMAN INTERVENTION
Student:	LUTOPU KHOA
Student Number:	201013380
Supervisor(s):	DR. MORGAN KAMGA-PENE DR. RODRIGUE GNITCHOGNA

Centre for Research Services

Take note of the following:

1. Any significant changes in the conditions or undertakings outlined in the approved Proposal must be communicated to the ethics committee. An application to make amendments may be necessary.
2. Any breaches of ethical undertakings or practices that have an impact on ethical conduct of the research must be reported to the ethics committee
3. The Principal Researcher must report issues of ethical compliance to the ethics committee (through the Chairperson) at the end of the Project or as may be requested by the ethics committee
4. The ethics committee retains the right to:

i) Withdraw or amend this Ethical Clearance if any unethical practices (as outlined in the Research Ethics Policy) have been detected or suspected,

ii) Request for an ethical compliance report at any point during the course of the research.

The ethics committee wishes you the best in your research.

Dr. Zivayi Chiguvare (Chairperson Ethics Committee)

Prof. Davis Mumbengegwi (Head, Multidisciplinary Research)

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RESEARCH PERMISSION LETTER

Date: 30/06/2022

Student Name: LUTOPU KHOA

Student Number: 201013380

Programme: Master of Science in Applied Mathematics

Approved Research Title: Analysis of Predator-Prey Models with Infection in Both the Predator and Prey and Human Intervention.

TO WHOM IT MAY CONCERN

I hereby confirm that the above-mentioned student is registered at the University of Namibia for the programme indicated. The proposed study met all the requirements as stipulated in the University guidelines and has been approved by the relevant committees.

The proposal adheres to ethical principles as per attached Ethical Clearance Certificate. Permission is hereby granted to carry out the research as described in the approved proposal.

Best Regards

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