NEIGHBOURHOOD DISTINGUISHING COLOURINGS OF GRAPHS

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Abstract

In this mini thesis, we study neighbourhood distinguishing colouring (NDC) of graphs, which are proper colourings of the vertices with the added condition that for every pair u, v of distinct vertices there is some colour c such that the number of vertices of colour c adjacent to u is different to the number of vertices of colour c adjacent to v. The neighbourhood distinguishing colouring number $\chi_{NDC}(G)$ is defined as the minimum cardinality of a neighbourhood distinguishing colouring of a graph G. The study begins with the discussion of some terminologies and definitions used later on in our study. Moreover, we consider the colour classes corresponding to an NDC and the neighbourhood distinguishing colouring number of certain familiar classes of graphs such as paths, cycles and trees. In addition, we classify graphs with neighbourhood distinguishing colouring number $\chi_{NDC}(G)$ equal to two up to

isomorphism. The chromatic number χ_G of graphs G with χ_{NDC} equal to two is also two. Finally, we characterize graphs whose χ_{NDC} coincides with the order of the graph. These graphs possess a unique $\chi_{NDC}-$partition and they are either complete graphs or union of vertex disjoint edges. A $\chi_{NDC}-$partition of a graph G is a partition of G with χ_{NDC} elements. The aim of this study is to give a considerable discussion of the neighbourhood distinguishing colouring and also to light the way for further research in the field of colourings.

Keywords: Neighbourhood distinguishing colouring, Neighbourhood distinguishing colouring partition, Neighbourhood distinguishing colouring number.

Contents

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My sincere thanks and appreciation to my co-supervisor Prof. Peter Dankelmann for his constant support, availability and constructive suggestions, which were determinant for the accomplishment of the work presented in this thesis.

Dedication

I dedicate this work to my beloved parents, who constantly provided their moral, emotional and financial support.

Declaration

I, Helena Nalitende Nafuka , hereby declare that this study, **Neighbourhood Distinguishing Colourings of graphs**, is my own work and is a true reflection of my research, and that this work, or any part thereof has not been submitted for a degree at any other institution.

Helena N. Nafuka **Signature:............** **January 2020**

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Chapter 1

Introduction

1.1 Background of the study

Distinguishing the vertices of a graph has been a topic of considerable interest in graph theory. In this context, Karpovsky, Chakrabarty and Levitin in 1998 introduced identifying codes to model fault diagnosis in multiprocessor systems [9]. An identifying code of a graph is a subset of its vertices such that every vertex of the graph is uniquely identified by the set of its neighbours within that subset. A number of mathematicians have worked on the topic, see [2, 14]. Motivated by these papers, a new type of coding called the Neighbourhood Distinguishing Colouring (*NDC*) codes is introduced. In this code, proper colour partitions of the vertex set are considered and the code does not involve the colour of the vertex but the number of neighbours having a prescribed colour is considered for defining the code. Several of the basic results on *NDC* colourings are included in this thesis.

1.2 Literature review

Several authors have considered the idea of distinguishing the vertices in a graph from one another through assigning codes that reflect the structure of the graph. In studying point determination in graphs, Sumner [13] and later Entringer and Gassman [4] considered the following question: Which graphs G have the property that for every pair u, v of vertices of G, $N(u) = N(v)$ implies that $u = v$? In such a graph, every vertex u can be distinguished from every other vertex by the map $u \mapsto N(u)$. These are precisely the graphs that have a neighbourhood distinguishing colouring [Theorem 2.3.1]. Majority of the existing literature has focused on the dimension/location approach to the problem of distinguishing the vertices, [5]. This approach was introduced separately by Harary and Melter [8] as well as Slater [12]. Each vertex of a connected graph G is distinguished from every other vertex of G by labelling an ordered subset S of V and using distances between the vertices of G and vertices of S to construct a one-to-one function on V. The symmetry breaking method was formalized by Albertson and Collins [1] and independently by Harary [7]. In this approach, a subset of the vertex set is coloured in such a way that the automorphism group of the graph is *destroyed*, i.e the automorphism group of the resulting structure is trivial. The difference worth noting between distance/location and symmetry breaking is that, in the former, a 1-1 function on $V(G)$ is usually explicitly present at the end of the process, while in the latter we are usually left only with an assurance that somehow every two vertices can be distinguished.

Code inducing vertex colouring and edge colouring have been studied by many. Identification of the vertices through codes determined by edge colourings has been studied by Escuadro and Zhang [6] . Creating codes for the vertices from vertex colouring which may not be a proper colouring has been adopted by Chartrand et al [2]. This colouring has been named recognizable colouring. If proper colourings are considered and the same type of coding for vertices as in recognizable colouring is adopted, irregular colourings are obtained [10]. Irregular colouring is a proper colouring which induces a distinguishing colour code for the vertices by means of the colour of the vertex represented by an integer and the number of neighbours of the vertex having a prescribed colour. A paper by Ramar and Venkatasubramanian [11] uses vertex colouring which is a proper colouring to create a code for vertices known as Neighbourhood Distinguishing Colouring (NDC) code. The code does not involve the colour of the vertex but the number of neighbours having a prescribed colour. Every graph has an irregular colouring, but not every graph has an NDC-colouring [10].

1.3 Thesis organization

This thesis is divided in six chapters. This being the very first chapter. Chapter two provides a brief account of the relevant definitions, existing results and examples pertaining to graph theory and Neighbourhood Distinguishing Colouring codes in particular. The third chapter presents the neighbourhood distinguishing colouring number (χ_{NDC}) for some well known graph classes such as paths, cycles and trees. The fourth chapter deals with graphs G for which $\chi_{NDC}(G) = 2$, where the characterization of this kind of graphs is provided. The fifth chapter investigates and characterizes graphs whose χ_{NDC} coincides with the number of vertices. The last chapter provides the summary and recommendations of the study.

Chapter 2

Preliminaries on Graph theory

2.1 Basic definitions and results

Definition 2.1.1. By a *simple graph* G, we mean a pair $G = (V, E)$ of a non-empty set V of *vertices* and a set E of two element subsets of V called the set of *edges*.

NB: Not all graphs are simple. Graphs containing multiple edges or directed edges or loops ($i.e\ \exists\ v \in V s.th\ vv\ \in E$) are not simple. In this paper, all graphs considered are simple unless otherwise stated.

Example 2.1.1. *Define a graph* G' *with* $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ *and* $E = \{\{u_1, u_2\},$ $\{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_4\}, \{u_3, u_4\}, \{u_1, u_5\}, \{u_4, u_6\}\}$.

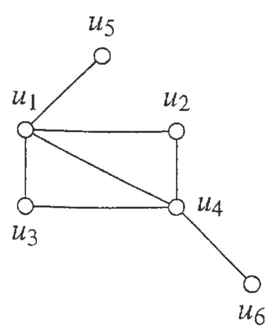


Fig. 2.1: A graph with 6 vertices and 7 edges

For notational convenience, instead of representing an edge as $\{u,v\}$, we denote it simply by uv. The *order* of a graph G is the number of vertices in its vertex set. Throughout this thesis, all graphs are assumed to have a finite vertex set.

Definition 2.1.2. Vertices u and v are *adjacent* (or *neighbours*) if $uv \in E$. The edge uv is *incident* with the vertices u and v For $u \in V$, the *neighbourhood* of u, denoted by $N(u)$, is given by:

$$N(u) = \{v \in V | uv \in E\}.$$

Definition 2.1.3. Let $G = (V,E)$ be a graph and $v \in V$. Then $G - v$ denotes the graph obtained by removing v and all edges incident with v from G.

Definition 2.1.4. Let G be a graph and v a vertex of G. The *degree* of v, denoted $deg(v)$, equals the number of edges that are incident to v. An *isolated vertex* is a vertex with degree zero. The *maximal degree* of a graph G, denoted by $\triangle(G)$, is defined as

$$\triangle(G) = max\{deg(v) | v \in V\}.$$

The total degree of G is the sum of the degrees of all the vertices of G.

The following fundamental lemma is called the "Handshake lemma" and can be found in every graph theory textbook [3] . We include the short proof.

Theorem 2.1.1. ***The Handshake Lemma.*** *Let $G = (V,E)$ be a graph. Then,*

$$\sum_{v \in V} deg(v) = 2|E|$$

Proof. Suppose that G has n vertices $v_1, v_2, ..., v_n$ and m edges, where n is a positive integer and m is a non-negative integer. We claim that each edge of G contributes 2 to the total degree of G. Suppose $v_i v_j$ is an arbitrarily chosen edge. This edge contributes 1 to the degree of v_i and 1 to the degree v_j. Therefore, v_iv_j contributes 2 to the total degree of G. Since v_iv_j was arbitrarily chosen, this shows that each edge of G contributes 2 to the total degree of G. Thus, the total degree of $G = 2$ (the number of edges of G). Thus,

$$\sum_{v \in V} deg(v) = 2|E|$$

□

Corollary 2.1.1. *A simple graph has an even number of vertices of odd degree*

Definition 2.1.5. Two edges e, e' are said to be *incident* if $|e \cap e'| = 1$.

Definition 2.1.6. A *walk* is a sequence $e_1, ..., e_t$ of edges such that e_i is incident to e_{i+1} for every $i \in \{1, ..., t-1\}$. A walk whose initial and terminal vertices are distinct is an *open walk*, otherwise is a closed walk.

Definition 2.1.7. An *Euler tour* is a closed walk containing every edge exactly once.

Definition 2.1.8. An *Eulerian graph* is a graph that is both connected and has a closed trail (walk with no repeated edges) containing all edges of the graph.

Definition 2.1.9. A *path* is a walk $e_1 = x_0x_1, ..., x_{n-1}x_n = e_n$ such that $x_i \neq x_j$ for $i \neq j$ where $i, j \in \{0, ..., n\}$. A path with n vertices is denoted by P_n. The length of a path is its number of edges. For $n = 4$, we have the following path of length three.

Fig. 2.2: The graph P_4

Definition 2.1.10. A graph G is said to be *connected* if for any two vertices x and y of G there is a path between x and y in G.

Definition 2.1.11. A graph $H = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. The following figure shows a subgraph of G'.

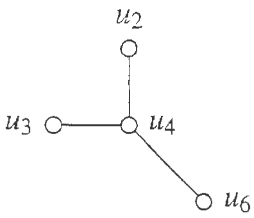


Fig. 2.3: Subgraph of G'

Definition 2.1.12. A subgraph $H = (V', E')$ is an induced subgraph of a graph $G = (V, E)$ if E' consists of all edges of G which have end points in V'. We write $H = G[V']$. The following figure shows an induced subgraph $(G'[u_1, u_3, u_4, u_5])$ and a subgraph of G' [on figure 2.1] which is not induced.

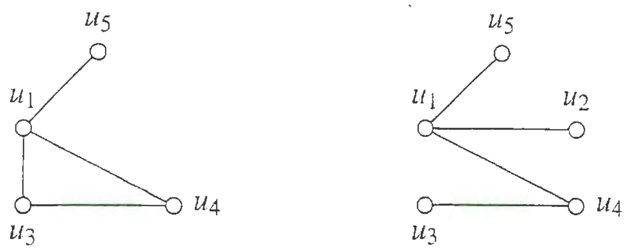


Fig. 2.4: Induced subgraph of G' and a subgraph of G' which is not induced

2.2 Special types of graphs

In this section we present several types of graphs that appear for our study.

1. Cycles

   A cycle is a graph $C = (V, E)$ where $v = \{x_1, ..., x_n\}$ with distinct vertices and $E = \{x_1x_2, ..., x_{n-1}x_n, x_nx_1\}, n \geq 3$. A cycle with n vertices is denoted by C_n. If $n = 5$, we have the following cycle.

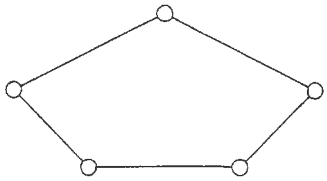


Fig. 2.5: The graph C_5

2. Trees

A tree, usually denoted by T is defined as a connected cycle-free graph.

A vertex of degree one in a tree is called a leaf.

Theorem 2.2.1. *Let T be a finite tree with $|V(T)| \geq 2$. Then, T has a leaf.*

Proof. Let $P = v_0v_1.....v_n$ be a longest path in T. Since P cannot be extended to a longer path, $N(v_n) \subseteq \{v_0, v_1, ..., v_{n-1}\}$. If $v_i \in N(v_n)$ for some $i < n-1$, then $v_iv_{i+1}........v_nv_i$ is a cycle in T, a contradiction since trees are cycle-free. Hence, the terminal (and the initial) vertex of a longest path in T have degree one. □

The following is a well known formula, but we provide the proof for the sake of completeness.

Theorem 2.2.2. *Let T be a tree with $|T| = n \geq 2$. Let n_i be the number of vertices of degree i in T. Then,*

$$n_1 = \sum_{i \geq 3}(i-2)n_i + 2$$

Proof. Since a tree of order n has $n-1$ edges and $n \geq 2$, it follows from the Handshake Lemma that:

$$\sum_{i \geq 1} in_i = 2(n-1) = 2\sum_{i \geq 1} n_i - 2,$$

so

$$\sum_{i\geq 1}(i-2)n_i+2=0,$$

and hence

$$(-1)n_1+0.n_2+\sum_{i\geq 3}(i-2)n_i+2=0$$

Which simplifies to

$$n_1=\sum_{i\geq 3}(i-2)n_i+2$$

□

We have the following significant result [3] for trees.

Theorem 2.2.3. *For a tree T, let n_1 denotes the number of leaves. Then, $n_1 \geq \triangle$.*

Proof. From theorem 2.2.2 we have that for $n \geq 2$,

$$\begin{aligned} n_1 &= \sum_{i\geq 3}(i-2)n_i+2 \\ &\geq (\triangle-2)n_\triangle+2 \\ &\geq \triangle \end{aligned}$$

Since $n_\triangle \geq 1$ □

3. Complements

The complement of a graph G denoted by $\overline{G}$ is the graph whose vertex set is the same as that of G and whose edge set consists of all the edges that are not present in G. The figure below shows the complement of a graph G' on figure 2.1.

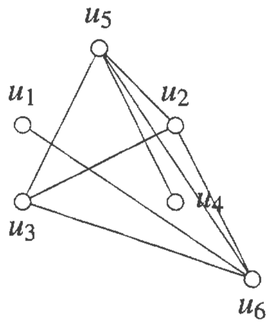


Fig. 2.6: The complement of G'

4. Complete Graphs

   A graph G is complete if every two distinct vertices in the graph are adjacent. The complete graph of order n is denoted by K_n. For $n = 6$, we have the following complete graph.

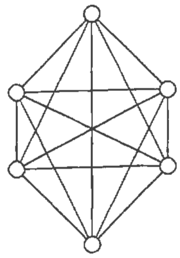


Fig. 2.7: The graph K_6

5. Bipartite Graphs

   A graph G is called a bipartite graph if it is possible to partition $V(G)$ into two non-empty subsets A and B , called partite sets, such that every edge of G joins a vertex of A and a vertex of B. The following figure shows a bipartite graph of order 6.

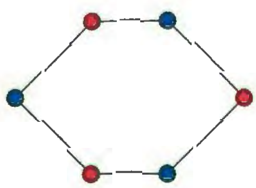
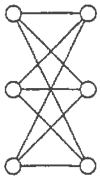


Fig. 2.8: Bipartite graph

A bipartite graph having partite sets A and B is a complete bipartite graph if every vertex of A is adjacent to every vertex of B. A complete bipartite graph with partite sets A and B containing r and s vertices respectively is denoted by $K_{r,s}$. The following graph is a complete bipartite graph.



A B

Fig. 2.9: Complete bipartite graph $K_{3,3}$

We close this section with the definition of isomorphism of graphs.

Definition 2.2.1. Let G and H be graphs. Then G and H are said to be *isomorphic* if there exists a one-to-one correspondence $f: V(G) \to V(H)$ such that for each pair u, v of vertices of G, $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

In simple terms, G and H are *isomorphic* if there exists a mapping from one vertex set to another that preserve adjacencies. The mapping is called an *isomorphism.*

Example 2.2.1. *The following graphs are isomorphic.*

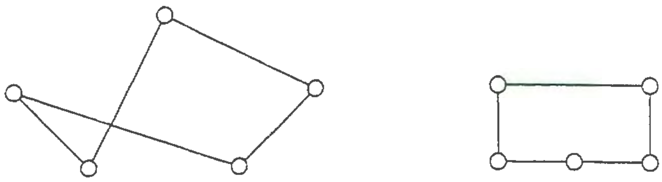


Fig. 2.10: Isomorphic graphs

2.3 Neighbourhood Distinguishing Colouring codes

Definition 2.3.1. Let $G = (V, E)$ be a simple connected graph. A proper vertex colouring of G is an assignment of colours to the vertices of G, one colour to each vertex so that adjacent vertices are coloured differently. A proper vertex colouring can be considered as a function $f\colon V \to \mathbb{N}$ such that $f(u) \neq f(v)$ if u and v are adjacent in G. If each colour used is one of k given colours, then we refer to the colouring as $k-$colouring. The chromatic number of G, denoted by $\chi(G)$, is the minimal k such that there is a proper vertex colouring of G with k colours. If $V_i(1 \leq i \leq k)$ is the set of vertices in G coloured i (where one or more of these sets may be empty), then each non-empty set V_i is called a colour class and the non-empty elements of $\{V_1, V_2, ..., V_k\}$ constitute a partition of V called a proper colour partition of V.

Definition 2.3.2. Let $G = (V, E)$ be a graph. Let $\pi = \{V_1, V_2, ..., V_n\}$ be a proper colour partition of V. Fixing this order of π, we define $C(v) = (|N(v) \cap V_1|, |N(v) \cap V_2|, \ldots, |N(v) \cap V_n|)$ for $v \in V$. The partition π is called an Neighbourhood Distinguishing Colouring (abbreviated as NDC)partition if $C(u) \neq C(v)$ whenever $u, v \in V$ and $u \neq v$. A graph is said to admit NDC if it has a neighbourhood distinguishing colouring.

The following theorem [11] characterizes graphs which possess an NDC colouring.

Theorem 2.3.1. *A graph G has NDC if and only if any two non-adjacent vertices of G do not have the same neighbourhood.*

Proof. Suppose G admits NDC. Let $\pi = \{V_1, V_2, ..., V_k\}$ be a $NDC-$partition. Let $u, v \in V(G)$ be distinct vertices. Then $C(u) \neq C(v)$. Therefore, there exists $1 \leq j \leq k$ such that $|N(u) \cap V_j| \neq |N(v) \cap V_j|$. In particular $N(u) \neq N(v)$.

Conversely, suppose for any two non-adjacent vertices u, v $N(u) \neq N(v)$. Let $\pi = \{\{x_1\}\{x_2\}, ..., \{x_k\}\}$ where $V(G) = \{x_1, ..., x_k\}$. If x_i and $x_j, 1 \leq i, j \leq k, i \neq j$ are adjacent then $C(x_i)$ has 0 in the i^{th} place and $C(x_j)$ has 1 in the i^{th} place. Therefore

$C(x_i) \neq C(x_j)$. If x_i and x_j are non-adjacent, then there exists x_k such that $x_k \in N(x_i)$ and $x_k \notin N(x_j)$ or vice versa. Hence $C(x_i) \neq C(x_j)$. Therefore, π is an *NDC* partition. $\square$

Example 2.3.1. *The following graph admits NDC.*

Let $H =$

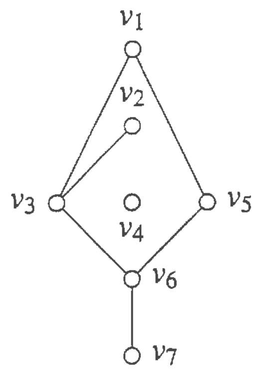


Fig. 2.11: NDC graph

Consider the following partition of H:

$\pi = \{\{v_1, v_4, v_2, v_6\}, \{v_3, v_5, v_7\}\}$

The codes for the vertices with respect to π are:

$$C_\pi(v_1) = (0,2),\ C_\pi(v_2) = (0,1),$$

$$C_\pi(v_3) = (3,0),\ C_\pi(v_4) = (0,0),$$

$$C_\pi(v_5) = (2,0),\ C_\pi(v_6) = (0,3),$$

$$C_\pi(v_7) = (1,0)$$

Clearly, the codes are distinct. Hence π is a $NDC-$partition of H.

Definition 2.3.3. Let G be a graph which admits NDC. The minimum cardinality of a neighbourhood distinguishing colouring of G is called the neighbourhood distinguishing colouring number of G, denoted by $\chi_{NDC}(G)$. A neighbourhood distinguishing colour partition of G with $\chi_{NDC}(G)$ elements is called a $\chi_{NDC}-$partition of G.

Remark 2.3.1. *If a graph G admits NDC, then $\chi(G) \leq \chi_{NDC}(G)$.* [11].

Example 2.3.2. *Consider the graph H in example 2.3.1. The chromatic number of H ($\chi(H)$) is two. By remark 2.3.1, $\chi_{NDC}(H) \geq 2$. Since π is an NDC−partition of H with $|\pi| = 2$, $\chi_{NDC}(H) = 2$*

Chapter 3

Neighbourhood distinguishing colouring numbers of some particular types of graphs

Graph classes such as paths, trees and cycles play important roles in many applications. This chapter discusses the neighbourhood distinguishing colouring number of the above mentioned graphs. Unfortunately, we have not been able to obtain sharp bounds. Our proofs show that those would be equivalent to answering some very difficult questions about recurrence patterns of vertices in Euler tours

3.1 Paths

The following result indicates that all paths except those of order three admits NDC.

Theorem 3.1.1. *A path on n vertices admits NDC if and only if $n \neq 3$*

Proof. Let $P = v_0v_1.....v_{n-1}$. If $n = 3$, then $P = v_0v_1v_2$.

$$N(v_0) = \{v_1\},$$

$$N(v_1) = \{v_0, v_2\},$$

$$N(v_2) = \{v_1\}$$

The non-adjacent vertices v_0, v_2 have the same neighbour. Therefore, by theorem 2.3.1, P_3 does not admit NDC. Now, suppose $n \neq 3$. Then,

$N(v_0) = \{v_1\}$ and $N(v_{n-1}) = \{v_{n-2}\}$; $1 \neq n-2$, so $N(v_0) \neq N(v_{n-1})$. If $1 \leq i, j \leq n-2, i \neq j$ then $N(v_i) = \{v_{i-1}, v_{i+1}\}$ and $N(v_j) = \{v_{j-1}, v_{j+1}\}$. $i \neq j$ implies that $j-1 \neq i-1$ thus $\{i-1, i+1\} \neq \{j-1, j+1\}$. This implies that non-adjacent vertices have distinct neighbourhoods and it follows from theorem 2.3.1 that $P_n, n \neq 3$ admits NDC.

□

Theorem 3.1.2. *Let P be a path on $n+1$ vertices and P has an NDC colouring with k colours. Then,*

$$n-1 \leq \binom{k+1}{2} - 2, \text{ if } k \text{ is odd and}$$

$$n-1 \leqslant \binom{k+1}{2} - k + 2, \text{ if } k \text{ is even}$$

Proof. Let $P = v_0, \ldots\ldots, v_n$ be a path of length n and let f be an NDC-colouring of the vertices of P with the colours $\{1, \ldots, k\}$. Let $a_i = f(v_i)$. The sets $\{a_i, a_{i+2}\}, i = 0, \ldots, n-2$, are pairwise distinct. So certainly $n-1 \leq \binom{k}{2} + k = \binom{k+1}{2}$. Moreover, $a_1 \neq a_0$ and $a_j \neq a_{j-1}$ if $j \geq 3$. Finally, $a_1 \neq a_{n-1}$. We define a graph

$$\Gamma = (V, E) \text{ where } V = \{a_0, \ldots, a_n\},\ E = \{a_i a_{i+2}, i = 0, \ldots., k-2\} \qquad (3.1.1)$$

If we direct the edges of Γ from a_i to a_{i+2} $(i = 0, \ldots., n-2)$ the in-degree of a_i, then for $i \notin \{0, 1, n-1, n\}$ the number of edges of Γ with two different vertices and a_i as initial vertex is equal to the number of edges with two different vertices and a_i as terminal vertex. Removing any loop aa from the edge set of Γ, we obtain a simple graph Γ', which has an even number of vertices of odd degree.

Suppose that k is odd. If $|\{a_0, a_1, a_{n-1}, a_n\}| = 4$, then Γ' has 4 vertices of odd degree and $n-1 \leq \binom{k+1}{2} - 2$. Suppose that $|\{a_0, a_1, a_{n-1}, a_n\}| = 3$. Let Γ_0' be the induced subgraph of Γ_0' on the even-numbered vertices and Γ_1' be the one on the odd-numbered vertices. Since both graphs have an even number of vertices of odd degree, $|\{a_0, a_1, a_{n-1}, a_n\}| = 3$ means that $a_0 = a_{n-1}$ if n is odd. Moreover, $n-1 \leq \binom{k+1}{2} - 2$. Suppose that $|\{a_0, a_1, a_{n-1}, a_n\}| = 2$. Since $a_0 \neq a_1 \neq a_{n-1}$, this means $a_0 = a_{n-1}$ and $a_1 = a_n$. In this case, all vertices in Γ' have even degree. This is the only case when $n-1 = \binom{k+1}{2}$ is possible.

If k is even, we obtain a much better bound, since the vertices of the complete graph K_k have odd degree. For even k, the bound is $n-1 \leqslant \binom{k+1}{2} - k + 2$, since $n-2$ vertices must have even degree. □

3.2 Cycles

The following result indicates that all cycles except those of order four admits *NDC*.

Theorem 3.2.1. *A cycle on n vertices admits NDC if and only if $n \neq 4$.*

Proof. Let $C = v_0 v_1 \dots\dots v_{n-1} v_0$. For $n = 4$, we have the cycle $C = v_0 v_1 v_2 v_3 v_0$. Then,

$$N(v_0) = \{v_1, v_3\},$$

$$N(v_1) = \{v_0, v_2\},$$

$$N(v_2) = \{v_1, v_3\},$$

$$N(v_3) = \{v_0, v_2\}$$

The non-adjacent vertices v_0, v_2 and v_1, v_3 have the same neighbourhood. Therefore, by theorem 2.3.1, C_4 does not admit NDC.

Suppose $n \neq 4$. Then, $N(v_0) = \{v_1, v_{n-1}\}$ and $N(v_{n-1}) = \{v_0, v_{n-2}\}, \{1, n-1\} \neq \{0, n-2\}$, so $N(v_0) \neq N(v_{n-1})$. If $1 \leq i, j \leq n-2, i \neq j$, then $N(v_i) = \{v_{i-1}, v_{i+1}\}$. $i-1 \neq j-1$ as $i \neq j$ and it follows that non-adjacent vertices do not have the same neighbourhoods hence from theorem 2.3.1 we have that $C_n, n \neq 4$ admits NDC. □

Theorem 3.2.2. *If C_n denotes a cycle on n vertices and C_n has an NDC-colouring with k colours, then $n \leq \binom{k+1}{2}$.*

Proof. Let $2 < n \in \mathbb{N}, C_n \cong C = v_0 v_1 .... v_{n-1} v_0$. We let $n \neq 4$ and $f : \{v_0, ..., v_{n-1}\} \rightarrow \{1, .., k\}$ be an $NDC-$colouring. As before, we let $a_i = f(v_i)$.

If n is odd, then $a_0a_2, a_1a_3, ..., a_{n-3}a_{n-1}, a_{n-2}a_1$ are the edges of a closed walk in Γ_k (3.1.1) . In particular, Γ_k is an Eulerian graph. Let Q_k be the complete graph on k vertices with an additional loop at every vertex. The graph Q_k is Eulerian if and only if K_k is i.e. if k is odd. So $n \leq \binom{k+1}{2}$, and equality is possible only if k is odd. However, $n = \binom{k+1}{2}$ would mean there is a subpath $v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}$ (vertices are labelled modulo n) of C_n coloured *cabad* with $d \neq c$. If we remove v_{i+2} and the incident edges from the graph, we obtain a path P on $n-1$ vertices with $n-1 \leq \binom{k+1}{2} - 2$. This follows from our analysis of paths with an $NDC-$colouring in which the initial and the terminal vertex are coloured the same.

If k is even, we obtain a better bound indeed, a longest closed walk in Γ_k has $\binom{k}{2}$ edges. If n is even, then the edges $a_0a_2, a_2a_4, ..., a_{n-4}a_{n-2}$ and $a_1a_3, ..., a_{n-3}a_{n-1}$ form (edge-disjoint) walks in Γ_k. □

Corollary 3.2.1. *Let C_n be a cycle on n vertices. Then $\chi_{NDC}(C_n) \geq l$ where l is minimal with $\binom{l+1}{2} \geq n$.*

Lemma 3.2.1. $\chi_{NDC}(C_6) = 5$ *and* $\chi_{NDC}(C_7) = 4$

Proof. $\chi_{NDC}(C_6) \geq 3$ as 3 is minimal with $\binom{3+1}{2} \geq 6$. Suppose $\chi_{NDC}(C_6) = 3$. Assume $f : \{v_1, v_2, v_3, v_4, v_5, v_6\} \to \{1,2,3\}$ is an *NDC* colouring of C_6. Let $C_6 = v_1v_2v_3v_4v_5v_6v_1$. Then $N(v_4) = \{v_3, v_5\}$. If we colour v_3 and v_5 with the same colour, then v_2 and v_6 will have neighbours with the same colouring. The same applies to the neighbours of v_1, v_2, v_3, v_5 and v_6. Therefore, an *NDC* colouring of C_6 assigns different colours to the neighbours of each vertex.

Lets colour v_1 with colour 1, v_2 with colour 2 and v_6 with colour 3. Then v_3 can only be coloured with colour 3 and v_5 with colour 2. If we colour v_3 and v_5 with this colours, then v_4 gets colour 1 and v_1, v_4 have neighbours with the same colouring. Therefore, f is not an *NDC* colouring and it follows that $\chi_{NDC}(C_6) \neq 3$

Suppose $\chi_{NDC}(C_6) = 4$. Assume $f : \{v_1, v_2, v_3, v_4, v_5, v_6\} \to \{1,2,3,4\}$ is an *NDC* colouring of $C_6 = v_1v_2v_3v_4v_5v_6v_1$. Lets colour v_1 with colour 1, v_2 with colour 2 and v_6 with colour 3. Since f is also a proper colouring, v_3 can only be coloured with colour 3 or 4 and v_5 can only be coloured with colour 2 or 4 . So, we have the following cases.

Case 1. If we colour v_3 with colour 3 and v_5 with colour 2, then v_4 gets colour 4 and the neighbours of v_1 and v_4 are coloured the same.

Case 2. If we colour v_3 with colour 3 and v_5 with colour 4, then v_4 gets colour 1 and the neighbours of v_2 and v_5 are coloured the same.

Case 3. If we colour v_3 with colour 4 and v_5 with colour 2, then v_4 gets colour 1 and the neighbours of v_3 and v_6 are coloured the same.

In all cases C_6 does not admit *NDC* and it follows that f is not an *NDC*$-$colouring. Therefore, $\chi_{NDC}(C_6) \neq 4$.

If we have 5 colours, then $\pi = \{\{v_1, v_4\}, \{v_2\}, \{v_3\}, \{v_5\}, \{v_6\}\}$ is an NDC−partition of C_6. Therefore, $\chi_{NDC}(C_6) = 5$.

For C_7, $\chi_{NDC}(C_7) \geq 4$ as 4 is minimal with $\binom{4+1}{2} \geq 7$. If we consider the proper colouring $f : \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \to \{1, 2, 3, 4\}$ where by $C_7 = v_1v_2v_3v_4v_5v_6v_7v_1$, then we see that $\pi = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5, v_7\}, \{v_6\}\}$ is an NDC−partition of C_7. Therefore, $\chi_{NDC}(C_7) = 4$. □

3.3 Trees

The following theorem helps to distinguish between trees that have *NDC* and those that do not.

Theorem 3.3.1. *Let T be a tree on n vertices, $n > 2$. Then T has NDC if and only if no two leaves of T have the same neighbour.*

Proof. Suppose T has NDC. Let u, v be two leaves of T. Then, u and v are non-adjacent hence by theorem 3.2.1, $N(u) \neq N(v)$. Thus, u and v do not have the same neighbour. Conversely, suppose $N(u) \neq N(v)$ for any two leaves u and v . Let u, v be non-leaves (inner vertices). If $N(u) = N(v)$, then there are two vertices x and y such that xu, yu, xv, yv are edges. These edges forms a cycle. This is a contradiction as a tree is cycle free. Therefore, $N(u) \neq N(v)$ and it follows that T has NDC. □

Example 3.3.1. *The first two trees have NDC while the last two do not.*

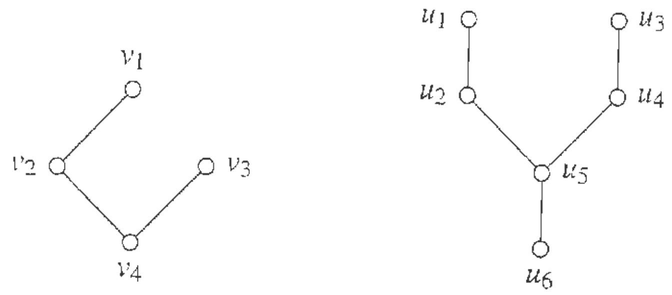


Fig. 3.1: NDC-Trees

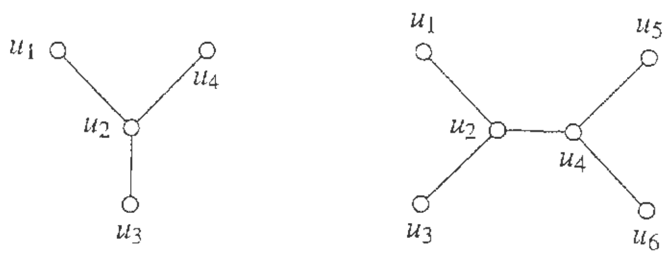


Fig. 3.2: Non-NDC Trees

The following result indicates the connection between trees that admit *NDC* and their χ_{NDC}.

Theorem 3.3.2. *If a tree T admits NDC, then $\chi_{NDC}(T) \geq n_1$, where n_1 denotes the number of leaves.*

Proof. Suppose a tree T admits *NDC*. First of all, note that two leaves in a tree on more than two vertices are not adjacent. If T has two vertices, then both vertices are leaves and $\chi_{NDC}(T) = 2$.

Now assume $|T| > 2$. Since T has an NDC-colouring no two leaves have the same neighbour by theorem 3.3.1. In an *NDC* colouring of T, the unique neighbours of each leaf need to have distinct colours. Therefore, the number of leaves of T is a lower bound of the neighbourhood distinguishing colouring number and thus the result follows.

□

Theorem 3.3.3. *If T is a tree of order n and $\chi_{NDC}(T) = k$, then*

$$n \leq \frac{1}{2}k^2 + \frac{5}{2}k - 2$$

Proof. Let $c : V(T_n) \to \{1, ..., k\}$ be an NDC-colouring with k colours. Let n_j be the number of vertices of degree i. By theorem 3.3.2,

$$n_1 \leq k \tag{3.3.1}$$

By theorem 3.3.2, if a vertex v has degree two, then either both neighbours of v have the same colour, so the code of v is $(0, ..., 0, 2, 0, ..., 0)$, or they have distinct colours, so the code of v is $(0, ..., 0, 1, 0..., 1)$. There are k possible codes of the former kind, and $\binom{k}{2}$ possible codes of the latter kind. Hence

$$n_2 \leq \binom{k}{2} + k \tag{3.3.2}$$

we used a similar argument in the "paths" section. Since in every tree we have

$$n_1 = \sum_{i \geq 3} (i-2)n_i + 2$$

It follows that

$$\begin{aligned} k \geq n_1 &= (\sum_{i \geq 3} (i-2)n_i) + 2 \\ &\geq (\sum_{i \geq 3} n_i) + 2 \end{aligned}$$

and so

$$\sum_{i \geq 3} n_i \leq k - 2 \tag{3.3.3}$$

In total, adding 3.3.1, 3.3.2, 3.3.3, we get

$$n = n_1 + n_2 + \sum_{i \geq 3} n_i \leq k + \binom{k}{2} + k + k - 2$$

and simplifying the right hand side yields the result. □

Example 3.3.2. *The following figure shows that the bound in Theorem* 3.3.3 *is sharp for* $k = 3$.

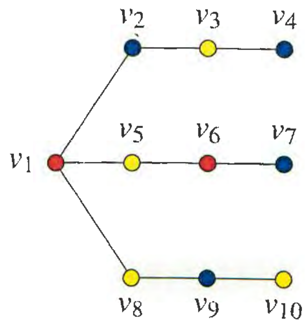


Fig. 3.3: A tree of order 10

Chapter 4

Characteristics of graphs with Neighbourhood distinguishing colouring number two

We classify the graphs G that have $\chi_{NDC}(G) = 2$ up to isomorphism. If G is a graph with $\chi_{NDC}(G) = 2$, then by remark 2.3.1, $\chi(G) \leq \chi_{NDC}(G) = 2$. The only graph G with $\chi(G) = 1$ and admitting NDC is the graph K_1 for which $\chi(G) = \chi_{NDC}(G) = 1$. So we are looking at graphs G with $\chi(G) = 2 = \chi_{NDC}(G)$, i.e. bipartite graphs in which there exists a bipartition such that the two colour classes form a neighbourhood distinguishing partition.

Lemma 4.0.1. *Let $G = (A \cup B, E)$ be a bipartite graph with $\chi_{NDC}(G) = 2$. Then the following hold.*

1. $||A| - |B|| \leq 1$

2. *The graph G has at most one isolated vertex. If G has one isolated vertex, then $||A| - |B|| = 1$, otherwise $|A| = |B|$.*

3. *Suppose that* $|A| = |B| = k$. *The vertices can be numbered such that* $N(u_i) = \{v_k, v_{k-1} ..., v_{k-i+1}\}, N(v_i) = \{u_k, u_{k-1} ..., u_{k-i+1}\}, 1 \leq i \leq k$ *and G is uniquely defined up to isomorphism.*

4. *Suppose that G has an isolated vertex u. Then* $G - u$ *is a graph as in 3.*

Proof. We start by proving statement one. Suppose $|A| \geq |B| + 2$. Let $A = \{u_1, u_2, ..., u_t\}$ and $B = \{v_1, v_2, ...v_s\}$. Since $\pi = \{A, B\}$ is a $NDC-$partition of G, the code of u_i is $(0, |N(u_i)|)$. Therefore, $|N(u_1)|, |N(u_2)|, ..., |N(u_t)|$ are all distinct. Suppose that $u_1, ..., u_t$ are arranged so that $d(u_1) < d(u_2) < ... < d(u_t)$. Then $d(u_t) > t - 1$. In particular, $|B| \geq t - 1$, i.e $s \geq t - 1$, a contradiction since $s \leq t - 2$. Therefore, $t \leq s + 1$ i.e $||A| - |B|| \leq 1$.

Next we prove statement two. It follows from Theorem 2.3.1 that a graph admitting NDC can have at most one isolated vertex. Suppose G has one isolated vertex. Without loss of generality, this is a vertex of A. If $w \in B$, then $0 < |N(w)| \neq |A|$ because w is not isolated and the isolated vertex in A cannot be adjacent to w. This means that $|N(w)| \in \{1, ..., |A| - 1\}$, i.e. $|B| = |A| - 1$. Therefore, $|A| - |B| = 1$ and the result follows.

Since $\chi_{NDC}(G) = 2$, $d(u) \neq d(v)$ for $u, v \in A$ and $d(x) \neq d(y)$ for $x, y \in B$. Since G has no isolated vertex, $d(v) \geq 1$ for all $v \in V$. For any $v \in A, d(v) \in \{1, 2, ..., |B|\}$, So $|A| \leq |B|$. Analogously, $|B| \leq |A|$, so $|B| = |A|$.

To prove statement 3, we let $|A| = |B| = k$ and label the vertices such that $d(u_i) = i = d(v_i)$ for all i. Since $d(u_k) = k$, u_k is adjacent to all vertices in B and since $d(u_{k-1}) = k - 1$, u_{k-1} is adjacent to all vertices in $B - \{v_1\}$. Vertex v_1 is left out simply because $d(v_1) = 1$, meaning it is only adjacent to one vertex which is v_k. Similarly, v_k is adjacent to all vertices in A and v_{k-1} is adjacent to all vertices in $A - \{u_1\}$.

We apply induction on i to show that $N(u_i) = \{v_k, \ldots v_{k-i+1}\}$ and $N(v_i) = \{u_k, \ldots, u_{k-i+1}\}$ whenever $1 \leq i \leq k$. We have seen that when $i = 1$, $N(u_1) = \{v_k\}$ and $N(v_1) = \{u_k\}$. Therefore, the result holds for $i = 1$. Suppose $N(u_i) = \{v_k, v_{k-1}, \ldots, v_{k-i+1}\}$, $N(v_i) = \{u_k, u_{k-1}, \ldots, u_{k-i+1}\}$ and for $1 \leq j \leq i$, $N(u_j) = \{v_k, \ldots, v_{k-j+1}\}$, $N(v_j) = \{u_k, \ldots, u_{k-j+1}\}$. Consider u_{i+1} and v_{i+1}. Then, $d(u_{i+1}) = i+1 = d(v_{i+1})$ and $d(u_{k-(i+1)+1}) = d(u_{k-i}) = k-i$. For $N(u_{i+1})$, consider $N(v_{k-i}) = \{u_k, \ldots\ldots, u_{k-(k-i)+1}\}$ $= \{u_{i+1}, \ldots\ldots, u_k\}$. By our inductive assumption, the vertices $u_1, \ldots, u_i$ do not belong to $N(v_{k-i})$. Pick a j in $\{1, \ldots, i\}$ and consider the neighbourhood of the vertex v_{k-j}. By assumption, the vertices $u_1, \ldots, u_j$ do not belong to $N(v_{k-j})$. Since $d(v_{k-j}) = k-j$, this means that $N(v_{k-j}) = \{u_{j+1}, \ldots, u_k\}$. In particular, u_{i+1} belongs to $N(v_{k-j})$. Since this is true whenever $1 \leq j \leq i$ and since $d(u_{i+1}) = i+1$, the neighbourhood of the vertex u_{i+1} is therefore the set $\{v_{k-i}, \ldots, v_k\}$ as claimed.

Analogously, for $N(v_{i+1})$ we consider $N(u_{k-i}) = \{v_k, \ldots\ldots, v_{k-(k-i)+1}\} = \{v_{i+1}, \ldots\ldots, v_k\}$. By our inductive assumption, the vertices $v_1, \ldots, v_i$ do not belong to $N(u_{k-i})$. Pick a j in $\{1, \ldots, i\}$ and consider the neighbourhood of the vertex u_{k-j}. By assumption, the vertices $v_1, \ldots, v_j$ do not belong to $N(u_{k-j})$. Since $d(u_{k-j}) = k-j$, this means that $N(u_{k-j}) = \{v_{j+1}, \ldots, v_k\}$. In particular, v_{i+1} belongs to $N(u_{k-j})$. Since this is true whenever $1 \leq j \leq i$ and since $d(v_{i+1}) = i+1$, the neighbourhood of the vertex v_{i+1} is therefore the set $\{u_{k-i}, \ldots, u_k\}$ as claimed. Therefore, the result holds for $i+1$.

Illustration if $k = 4$, we have the following graph.

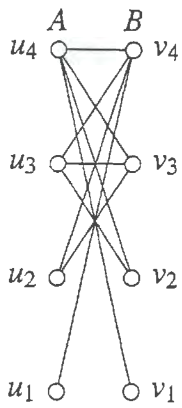


Fig. 4.1: Graph G for $k=4$

To show that graphs $G=(A\cup B,E)$ with $\chi_{NDC}(G)=2$ and $|A|=|B|$ are uniquely defined up to isomorphism, we let $G=(A\cup B,E)$ and $G'=(A'\cup B',E')$ be two graphs and let $\phi : A\cup B \rightarrow A'\cup B'$ be defined as follow: If $(u_i)_{1\leq i\leq k}\in A$ then $\phi(u_i)$ is the only vertex in A' of degree $d(u_i)$ and if $(v_j)_{1\leq j\leq i\leq k}\in B$, then $\phi(v_j)$ is the only vertex in B' of degree $d(v_j)$.

Now, suppose $u_iv_j\in E(G)$. Then,

$$N(u_i)=\{v_k,v_{k-1},\ldots,v_j,..,v_{k-i+1}\}$$

$$N(v_j)=\{u_k,u_{k-1},\ldots,u_i,..,u_{k-j+1}\}$$

$$N(\phi(u_i))=\{\phi(v_k),\phi(v_{k-1}),\ldots,\phi(v_j),..,\phi(v_{k-i+1})\}$$

$$N(\phi(v_j))=\{\phi(u_k),\phi(u_{k-1}),\ldots,\phi(u_i),..,\phi(u_{k-j+1})\}$$

This implies that $\phi(u_i)\phi(v_j)\in E(G')$.

Conversely, suppose $\phi(u_i)\phi(v_j)\in E(G')$. Then $\phi(u_i)\in N(\phi(v_j))$ and $\phi(v_j)\in N(\phi(u_i))$. Therefore $\phi^{-1}(\phi(u_i))=u_i\in N(\phi^{-1}(\phi(v_j)))=v_j$ and $\phi^{-1}(\phi(v_j))=v_j\in N(\phi_{-1}(\phi(u_i)))=u_i$. Hence, $u_iv_j\in E(G)$.

For the last statement, suppose G has an isolated vertex u. Then consider $G-u=G'$. Then $\chi NDC(G')=2$ and G' does not have an isolated vertex. Thus, if A and B are the colour classes of G', then $|A|=|B|$. Therefore, G' is exactly like the graph described in statement three □

Chapter 5

Characteristics of graphs with neighbourhood distinguishing colouring number equal to the order of the graph

In this chapter we characterize graphs whose χ_{NDC} coincides with the number of vertices.

We begin with the following observation which allows to assemble graphs with a given χ_{NDC} from smaller components.

Theorem 5.0.4. *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be NDC graphs on disjoint vertex sets V_1 and V_2. Let $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv | u \in V_1, v \in V_2\}$, we obtain the join graph usually denoted by $G + H = (V, E)$. Then $\chi_{NDC}(G+H) = \chi_{NDC}(G) + \chi_{NDC}(H)$.*

Proof. We first show that $G+H$ has an $NDC-$colouring. Suppose f is an $NDC-$colouring of G with the colours $\{1, 2, ..., k\}$ and g is an $NDC-$colouring of H with the colours

$\{1,2,...,l\}$. We define a colouring $g' : H \rightarrow \{k+1,k+2,...,k+l\}$ by $g'(v) = g(v)+k$. Now colour the vertices of $G+H$ by $h : V(G+H) \rightarrow \{1,2,...,k+l\}$ defined by

$$h(v) = \begin{cases} f(v) & v \in V(G) \\ g'(v) & v \in V(H) \end{cases}$$

This is a proper colouring because both f and g' are proper colourings and the respective colour sets are disjoint. Let u, v be vertices of $G+H$. Let the sets $V_1,...,V_{k+l}$ be the colour classes. If $V(G) = V_1 \cup ... \cup V_k, V(H) = V_{k+1} \cup ...... \cup V_{k+l}$, then

$$(|N(u) \cap V_1|,...,|N(u) \cap V_k|) \neq (|N(v) \cap V_1|,...,|N(v) \cap V_k|)$$

So,

$$(|N(u) \cap V_1|,...,|N(u) \cap V_k|, |V_{k+1}|,...,|V_{k+l}|) \neq$$

$$(|N(v) \cap V_1|,...,|N(v) \cap V_k|, |V_{k+1}|,...,|V_{k+l}|)$$

If $w \in H$, we get the code

$$(|V_1|,...,|V_k|, |N(w) \cap V_{k+1}|,...,|N(w) \cap V_k|)$$

If $u \in V(G)$ and $v \in V(H)$ then $N(u) \cap H = V(H)$ and $N(v) \cap H \subset V(H)$, so the two vertices have different codes and it follows that the new colouring is an $NDC-$colouring and $\chi_{NDC}(G+H) \leq \chi_{NDC}(G) + \chi_{NDC}(H)$.

Now we show that $\chi_{NDC}(G+H) \geq \chi_{NDC}(G) + \chi_{NDC}(H)$. In $G+H$, every vertex of G is a

neighbour of every vertex of H. So, a proper colouring of $G+H$ has to use disjoint colour sets for G and H. Given an $NDC-$colouring of K with m colours, the colour classes will always be of the form $W_1,...,W_m$ such that there is $1 < i < m$ so that

$W_1 \cup \ldots \cup W_i = V(G)$ and $W_{i+1} \cup \ldots \cup W_m = V(H)$. Now let $u, v \in V(G)$. The codes of the vertices u and v coming from the partition $W_1, \ldots, W_m$ are:

$$\text{for u: } (|N(u) \cap W_1|, \ldots, |N(u) \cap W_i|, |W_{i+1}|, \ldots, |W_m|)$$

$$\text{for v:}(|N(v) \cap W_1|, \ldots, |N(v) \cap W_i|, |w_{i+1}|, \ldots, |W_m|)$$

$$\text{So } (|N(u) \cap W_1|, \ldots, |N(u) \cap W_i|) \neq (|N(v) \cap W_1|, \ldots, |N(v) \cap W_i|)$$

Analogously, if $x, y \in V(H)$

$$(|N(x) \cap w_{i+1}|, \ldots, |N(x) \cap w_m|) \neq (|N(y) \cap w_{i+1}|, \ldots, |N(y) \cap w_m|)$$

So the colour classes $w_1, .., w_i$ form an $NDC-$partition for G and the colour classes $w_{i+1}, \ldots, w_m$ form an $NDC-$partition for H. So $\chi_{NDC}(K) = m = i + m - i \geq \chi_{NDC}(G) + \chi_{NDC}(H)$. Since $\chi_{NDC}(G+H) \leq \chi_{NDC}(G) + \chi_{NDC}(H)$ and $\chi_{NDC}(K) \geq \chi_{NDC}(G) + \chi_{NDC}(H)$, then $\chi_{NDC}(G+H) = \chi_{NDC}(G) + \chi_{NDC}(H)$ □

Consider a graph $G = (V, E)$ with $|V| = n = \chi_{NDC}(G)$. If $n = 2$, then $G = K_2$. We assume $n > 2$ from now on. Since $\chi_{NDC}(G) = n$, the only NDC colouring of G is the one where a different colour is assigned to each vertex. Equivalently, the only $NDC-$partition of G is the one into sets $\{v\}$, where $v \in V$. Therefore, graphs G with $\chi_{NDC}(G) = |V|$ admit a unique $\chi_{NDC}-$partition.

We have the following characterization of this kind of graphs.

Theorem 5.0.5. Let $G = (V, E)$ be a graph on n vertices. If $\chi_{NDC}(G) = n$, then there is a partition of V into non-empty subsets $V_1, \ldots., V_k$ such that:

1. For $i = 1, \ldots, k$, the induced subgraph $G[V_i]$ either has a single vertex or is the union of vertex-disjoint edges.

2. If $s \in V_i, t \in V_j, i \neq j, i, j \in \{1, \ldots, k\}$, then $st \in E$.

Proof. Suppose $\chi_{NDC}(G) = n$. If G is complete, then there is nothing left to prove. Otherwise, there are $u, v \in V$ with $u \neq v$ and $uv \notin E$. Let $f : V \to \{1, ..., n-1\}$ be a proper vertex colouring of G with $f(u) = f(v) = 1$ and $1 \neq f(w) \neq f(w')$ whenever $w, w' \in V \setminus \{u, v\}, w \neq w'$. Since f is not neighbourhood distinguishing, there are vertices x and y such that $xu \in E \not\ni yu$ and $yv \in E \not\ni xv$, so that $d(x) = d(y)$ and $\{u, v\} = N(x) \triangle N(y)$ ($\triangle$ denotes the symmetric difference of two sets). Since u and v are not adjacent, $\{x, y\} \cap \{u, v\} = \emptyset$; also $xy \notin E$ otherwise $x \in N(x) \triangle N(y)$. Let $S = V^{(2)} \setminus E$ and let T be the subset of S consisting of sets $\{x, y\}$ with $xy \notin E, d(x) = d(y)$ and $N(x) \triangle N(y) \in S$. For every $\{x, y\} \in T$, the set $N(x) \triangle N(y)$ is uniquely determined and every $\{u, v\} \in S$ is equal to $N(x) \triangle N(y)$ for some element $\{x, y\}$ of T. This is possible only if $S = T$, i.e

If $x, y \in V$ and $xy \notin E$, then $d(x) = d(y)$ and $N(x) \triangle N(y)$ consists of two non-adjacent vertices (*)

Let u, v, x, y be as before and let $N = N(x) \cap N(y)$. Letting $d = d(u)$, we have seen that $d = d(x) = d(y)$. Since $yu \notin E$ and $\{x, v\} \subset N(y) \triangle N(u)$, remark (*) yields that $\{x, v\} = N(y) \triangle N(u)$; hence $d(x) = d(v)$ and $N = N(x) \cap N(v)$. In the same way, we obtain $d(y) = d(u) = d(x)$ and $N = N(x) \cap N(v) \cap N(u) \cap N(y)$. Let $z \in W = V \setminus (N \cup \{u, v, x, y\})$. Then z is not adjacent to any of the vertices u, v, x, y, in particular $x \in N(u) \setminus N(z)$. By (*), $d(z) = d$ and $N \subset N(z)$. It follows that $N(z) \setminus N = \{w\}$ with $w \in W$. Now let $V_1 = W \cup \{u, v, x, y\}$. We have seen that the induced subgraph $G[V_1]$ is the union of $\frac{|V_1|}{2}$ vertex- disjoint edges. Let $V_2 = V \setminus V_1$. If $s \in V_1$ and $t \in V_2$ then $st \in E$. □

Remark 5.0.1. *Note that the graphs of theorem* 5.1.0 *can alternatively be described as follows:* $V(G)$ *is a disjoint union* $V(G) = W_1 \cup \ldots \cup W_\ell$ *of nonempty sets,* $G[W_i]$ *is* *either a complete graph or a union of vertex-disjoint edges and if* $1 \leq i < j \leq \ell$, $x \in W_i$ *and* $y \in W_j$, *then* $xy \in E$

The next result shows that complete graphs and unions of vertex-disjoint edges have neighbourhood distinguishing colouring number equal to the order of the graph.

Lemma 5.0.2. *Let* $G = (V,E)$ *be a connected graph. If* G *is a complete graph or* G *is a union of vertex-disjoint edges then* $\chi_{NDC}(G) = |V|$.

Proof. If G is complete then every vertex of G is adjacent with all the other vertices of G. Hence, the only $NDC-$colouring of G is the colouring where each vertex gets a different colour and it follows that $\chi_{NDC}(G) = |V|$.
If G is a union of vertex-disjoint edges, we have $|N(v)| = 1$ for $v \in V$. Also, for every vertex $u \in V$, $\{u\} = N(w)$ for some $w \in V$. The only $NDC-$colouring of G is the one in which every vertex gets a different colour. Therefore, $\chi_{NDC}(G) = |V|$. □

If we combine theorem 5.0.1 and *lemma*5.1.1, we can immediately deduce the following result.

Corollary 5.0.1. *The graphs (in theorem* 5.1.1*) of the form* $G = (V,E)$ *where* $V = V_1 \sqcup .... \sqcup V_k$ *for each* $i \in 1,..,k$ *such that* $G[V_i]$ *is complete or a disjoint union of edges and if* $i \neq j, x \in V_i, y \in V_j$*, the* $xy \in E$*, have* χ_{NDC} *equal to the number of vertices.*

Proof. We prove by induction on k. If $k = 1$, then G is complete or a union of vertex-disjoint edges and this graphs have χ_{NDC} equal to the number of vertices by *lemma*5.1.1. Assume it is true for $k-1$. Let $H = G[V_1 \cup ... \cup V_{k-1}]$ and $H' = G[V_k]$. By induction $\chi_{NDC}(H) = |V(H)|$. By *lemma*5.1.1, $\chi_{NDC}(H') = |V(H')|$. Now by theorem

5.0.1 we have that

$$\chi_{NDC}(G) = \chi_{NDC}(H) + \chi_{NDC}(H')$$

$$= |V(G)|$$

□

Chapter 6

Conclusion and Recommendations

6.1 Conclusion

In this thesis, we have studied the neighbourhood distinguishing colourings. A graph G is said to have NDC if and only if any two non-adjacent vertices of G do not have the same neighbourhood [11]. This result has shown success in establishing whether a graph has NDC or not. The result indicates that all paths except those of order three, all cycles except those of order four and trees whose any two leaves have distinct neighbors admit NDC. For some of this graphs that admit NDC, we managed to approximate their χ_{NDC}. Moreover, the chromatic number χ_G of graphs G with χ_{NDC} equal to two is also two. If G is a bipartite graph, we have shown that the cardinalities of the partite sets of G are the same provided G has no isolated vertex. If G has one isolated vertex then the absolute value of the difference of the cardinalities of the partite sets is one. We have also pointed out that graphs with χ_{NDC} equal to their order possess a unique $\chi_{NDC}-$partition. These graphs are either complete graphs or union of vertex disjoint edges.

6.2 Recommendations

The study intended to lay the groundwork for further studies in the neighbourhood distinguishing colouring. The study recommends further research on the neighbourhood distinguishing colouring number of other types of graphs apart from paths, cycles and trees.

Bibliography

[1] Albertson M. O. and Collins K.L., *Symmetry breaking in graphs*, Electronic Journal of Combinatorics, Proceedings of Royal Society of London, vol. 3, (1996).

[2] Chartrand G., Johnson M. and Oellermann O.R., *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl.Math, vol. 105, pp. 99–113 (2000).

[3] Diestel R., *Graph Theory*, 3rd edition, Springer-Verlag, Berlin and Heidelberg, vol. 3, 2005.

[4] Entringer R.C. and Gassman L.D., *Line-critical point determining and point distinguishing graphs*, Discrete Math, vol. 10, 1974, pp. 43–55.

[5] Erwin and Harary, *Destroying automorphisms by fixing nodes*, Discrete Mathematics, vol. 306, pp. 3244–3252 (2006).

[6] Escuadro H. and Zhang P., *Extremal problems on detectable colorings; of connected graphs with cycle rank 2.*, AKCE Int. J. Graphs Comb., vol. 2, pp. 99–117 (2005).

[7] Harary F., *Methods of destroying the symmetries of a graph*, Bull. Malaysian Math. Sci. Soc., vol. 24(2), pp. 183–191 (2001).

[8] Harary F. and Melter R.A., *On the metric dimension of a graph*, Ars Combin., vol. 2, pp. 191–195 (1997).

[9] Karpovsky M.G.; Chakrabarty K. and Levitin L.B., *On a new class of codes for identifying vertices in graphs*, IEEE Trans. Inform. Theory, vol. 44(2), pp. 599–611 (1998).

[10] Radcliffe M. and Zhang P., *Irregular colourings of graphs*, Bull. Inst. Combin. Appl., vol. 49, pp. 41–59 (2007).

[11] Ramar R. and Venkatasubramanian S., *Neighborhood distinguishing coloring in graphs*, Innovations in Incidence Geometry, vol. 13, pp. 135–140 (2013).

[12] Slater P.J., *Leaves of trees*, Congr. Numer., vol. 14, pp. 549–559 (1975).

[13] Sumner D.P., *Point determination in graphs*, Discrete Math, vol. 5, pp. 179–187 (1973).

[14] Sunganthi S., *A study on Resolving sets in Graphs*, Thesis submitted to Madurai Kamaraj University.